

A DENSITY THEOREM ON EVEN FAREY FRACTIONS

CRISTIAN COBELI AND ALEXANDRU ZAHARESCU

ABSTRACT. Let \mathfrak{F}_Q be the Farey sequence of order Q and let $\mathfrak{F}_{Q,\text{odd}}$ and $\mathfrak{F}_{Q,\text{even}}$ be the set of those Farey fractions of order Q with odd, respectively even denominators. A fundamental property of \mathfrak{F}_Q says that the sum of denominators of any pair of neighbor fractions is always greater than Q . This property fails for $\mathfrak{F}_{Q,\text{odd}}$ and for $\mathfrak{F}_{Q,\text{even}}$. The local density, as $Q \rightarrow \infty$, of the normalized pairs $(q'/Q, q''/Q)$, where q', q'' are denominators of consecutive fractions in $\mathfrak{F}_{Q,\text{odd}}$, was computed in [10]. The density increases over a series of quadrilateral steps ascending in a harmonic series towards the point $(1, 1)$. Numerical computations for small values of Q suggest that such a result should rather occur in the even case, while in the odd case the distribution of the corresponding points appears to be more uniform. Reconciling with the numerical experiments, in this paper we show that, as $Q \rightarrow \infty$, the local densities in the odd and even case coincide.

1. INTRODUCTION

Questions concerned with Farey sequences have a long history. In some problems, such as for instance those related to the connection between Farey fractions and Dirichlet L -functions, one is lead to consider subsequences of Farey fractions defined by congruence constraints. Recently it has been realized that knowledge of the distribution of subsets of Farey fractions with congruence constraints would also be useful in the study of the periodic two-dimensional Lorentz gas. This is a billiard system on the two-dimensional torus with one or more circular regions (scatterers) removed (see [21], [8], [9], [7]). Such systems were introduced in 1905 by Lorentz [20] to describe the dynamics of electrons in metals. A problem raised by Sinaï on the distribution of the free path length for this billiard system, when small scatterers are placed at integer points and the trajectory of the particle starts at the origin, was solved in [5], [6], using techniques developed in [1], [2], [3] to study the local spacing distribution of Farey sequences.

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The more general case when the trajectory starts at a given point with rational coordinates is intrinsically connected with the problem of the distribution of Farey fractions satisfying congruence constraints. For example, the case when the trajectory starts from the center $(1/2, 1/2)$ of the unit square is related to the distribution of Farey fractions with odd numerators and denominators. The distribution of the free path length computed in [5] and [6] is totally different from the one obtained in [7]. This confirms the intuition of physicists that, unlike in the case when the trajectory starts at the origin, if one averages over the initial position of the particle, the distribution will have a tail. It is then reasonable to expect that, in terms of the distribution of Farey fractions, new phenomena would be encountered when one replaces the entire sequence of Farey fractions by a subsequence defined by congruence constraints.

As we shall see below, already the case of the subsequence of Farey fractions with even denominators presents nontrivial complications. This is mainly due to the fact that in \mathfrak{F}_Q there is a large number of tuples of consecutive fractions with odd denominators and length growing to infinity with Q . Recently some questions on the distribution of Farey fractions with odd denominators have been investigated in [4], [10], and [19]. It is the purpose of this work to derive a result on fractions with even denominators.

Two fundamental properties of the Farey sequence of order Q state that if $a'/q' < a''/q''$ are consecutive elements of \mathfrak{F}_Q , then $a''q' - a'q'' = 1$, and $q' + q'' > Q$. These properties play an essential role in questions concerned with the distribution of Farey fractions. In fact, in any problem where one has an element a'/q' of \mathfrak{F}_Q and needs to find the next element of \mathfrak{F}_Q , call it a''/q'' , one can use the above two properties in order to determine a''/q'' , as follows. The equality $a''q' - a'q'' = 1$ uniquely determines a'' in terms of a' , q' and q'' . In order to find q'' in terms of a' and q' , notice from the above equality that $a'q'' \equiv -1 \pmod{q'}$. The inequalities $q' + q'' > Q$ and $q'' \leq Q$ show that q'' belongs to the interval $(Q - q', Q]$, which contains exactly one integer from each residue class modulo q' . Only one of these integers satisfies the congruence $a'q'' \equiv -1 \pmod{q'}$, and this uniquely determines q'' in terms of q' and a' . Complications arise when one studies a subsequence of Farey fractions with denominators in a given residue class modulo an integer number $d \geq 2$, since in such a case the above two properties fail (see [4], [10], [19] for the case of fractions with odd denominators).

In the present paper we study the relative size of consecutive even denominators in Farey series. Although the inequality $q' + q'' > Q$ fails in this case too, we shall see that

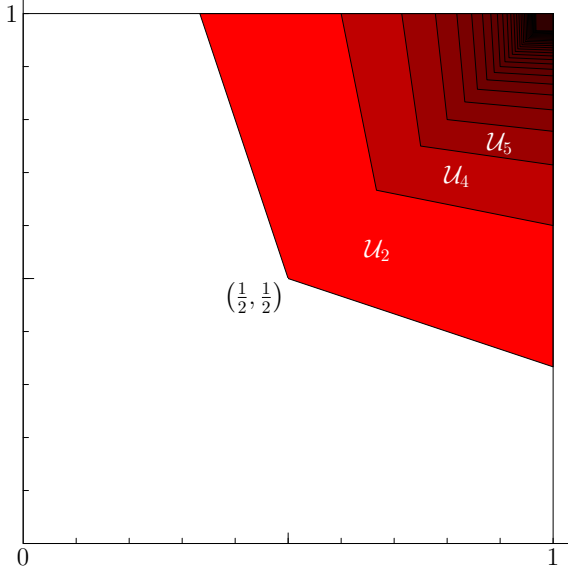


Figure 1: From light to dark are represented the sets of type $\mathcal{T}(1)$.

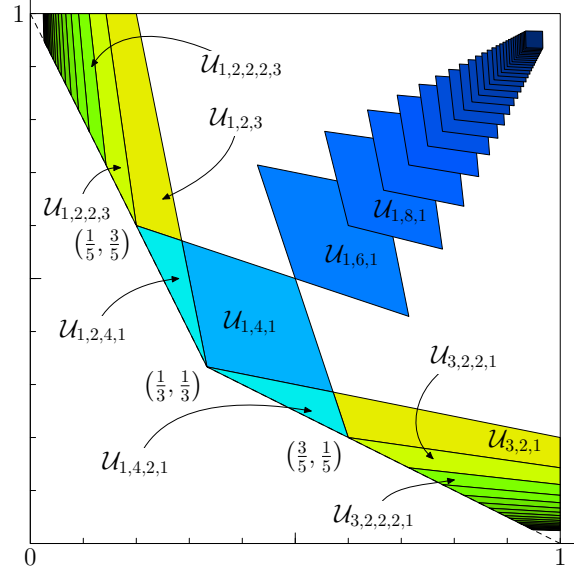
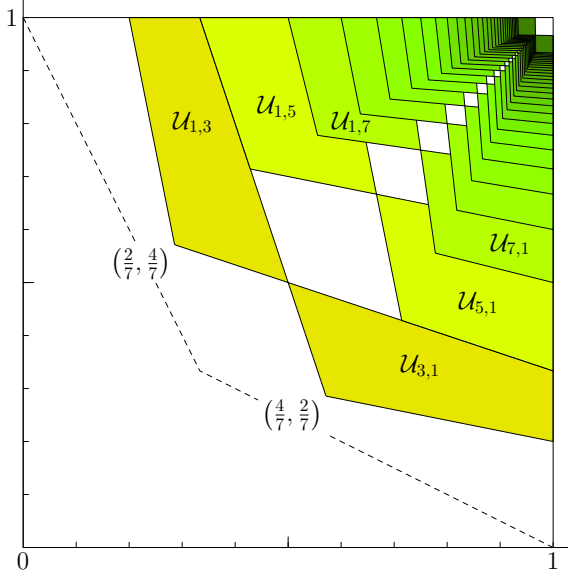
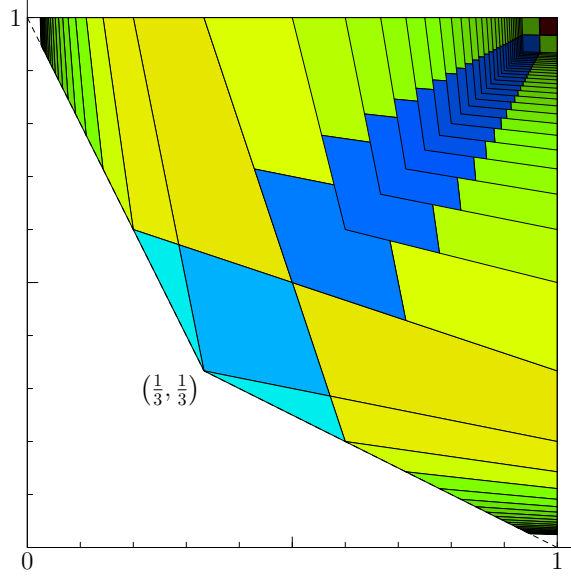


Figure 2: The sets of type $\mathcal{T}(3)$, $\mathcal{T}(4)$ and $\mathcal{T}(r)$, with $r \geq 5$.

the points $(q'/Q, q''/Q)$ have a limiting distribution inside the unit square $[0, 1] \times [0, 1]$, as $Q \rightarrow \infty$. In the following we let

$$\mathfrak{F}_{Q,\text{even}} = \left\{ \frac{a}{q} : 1 \leq a \leq q \leq Q, \gcd(a, q) = 1, q \equiv 0 \pmod{2} \right\}$$

and we always assume that the elements of $\mathfrak{F}_{Q,\text{even}}$ are arranged in increasing order. We call a Farey fraction *odd* if its denominator is odd and *even* if its denominator is even, respectively. A new feature that the sequence $\mathfrak{F}_{Q,\text{even}}$ brings in this type of problems comes from the following phenomenon. We know from the equality $a''q' - a'q'' = 1$ that the denominators q', q'' of any two consecutive fractions $a'/q' < a''/q''$ in \mathfrak{F}_Q are relatively prime, and in particular not both of them are even. Therefore, if we study the subset $\mathfrak{F}_{Q,\text{odd}}$ of odd Farey fractions in \mathfrak{F}_Q , we know that any two consecutive elements of $\mathfrak{F}_{Q,\text{odd}}$ are either consecutive in \mathfrak{F}_Q , or there is exactly one element of $\mathfrak{F}_{Q,\text{even}}$ between them. Thus there are only two types of situations to consider. By contrast, we may have any number of elements from $\mathfrak{F}_{Q,\text{odd}}$ between two consecutive elements of $\mathfrak{F}_{Q,\text{even}}$. This more complicated context that arises in the even case, treated in the present paper, forces us to go through a significantly larger amount of data than in the odd case. This is also reflected in the larger variety of situations that appear in Figures 1-4 and Tables 1-2 below, which show various aspects of the distribution in this case. In what follows we study the local density

Figure 3: The sequences of sets of type $T(2)$.Figure 4: The covering of $\mathcal{D}(0,2)$ by the sets of all types.

of points $(q'/Q, q''/Q)$, with q', q'' denominators of consecutive Farey fractions in $\mathfrak{F}_{Q,\text{even}}$, which lie around a given point (u, v) in the unit square. We shall show that this local density approaches a certain limit $g(u, v)$ as $Q \rightarrow \infty$, and we provide an explicit formula for $g(u, v)$.

The property of Farey fractions that first drew attention two centuries ago was their very uniform distribution of in $[0, 1]$ (see the survey paper [11] and the references therein). It is natural to expect various distribution results, in particular the one obtained in the present paper, to continue to hold in subintervals of $[0, 1]$. Let $\mathcal{I} \subseteq [0, 1]$ be a fixed subinterval, and denote $\mathfrak{F}_Q^{\mathcal{I}} = \mathfrak{F}_Q \cap \mathcal{I}$ and $\mathfrak{F}_{Q,\text{even}}^{\mathcal{I}} = \mathfrak{F}_{Q,\text{even}} \cap \mathcal{I}$. We let

$$\mathcal{D}_{Q,\text{even}}^{\mathcal{I}} := \{(q', q'') : q', q'' \text{ denominators of consecutive fractions in } \mathfrak{F}_{Q,\text{even}}^{\mathcal{I}}\},$$

and consider the set $\mathcal{D}_{Q,\text{even}}^{\mathcal{I}}/Q$, which is a subset of the unit square $[0, 1] \times [0, 1]$. If \mathcal{I} is the complete interval $[0, 1]$, we drop the superscript and write $\mathcal{D}_{Q,\text{even}} = \mathcal{D}_{Q,\text{even}}^{[0,1]}$.

For each point $(u, v) \in [0, 1] \times [0, 1]$, we take a small square \square centered at (u, v) and count the number of points from $\mathcal{D}_{Q,\text{even}}^{\mathcal{I}}/Q$ which lie inside the square \square . We shall see that, as $Q \rightarrow \infty$, the proportion of points from $\mathcal{D}_{Q,\text{even}}^{\mathcal{I}}/Q$ that fall inside \square approaches a certain limit. This limit will be proportional to the area of the square \square . After dividing this limit by $\text{Area}(\square)$ and letting the side of the square \square tend to 0, we arrive at a limit, call it $g^{\mathcal{I}}(u, v)$, which will only depend on the point (u, v) , and possibly on the interval \mathcal{I} .

Thus, we let

$$g^{\mathcal{I}}(u, v) := \lim_{\substack{Q \rightarrow \infty \\ \text{Area}(\square) \rightarrow 0}} \frac{\frac{\#(\square \cap \mathcal{D}_{Q, \text{even}}^{\mathcal{I}}/Q)}{\#\mathcal{D}_{Q, \text{even}}^{\mathcal{I}}}}{\text{Area}(\square)}, \quad (1)$$

in which $\square \subset \mathbb{R}^2$ are squares centered at (u, v) . We put $g(u, v) = g^{[0,1]}(u, v)$. Theorem 1 below shows that the limiting local density function $g(u, v)$ exists, and its value is calculated explicitly.

Since the Farey fractions are distributed in $[0, 1]$ symmetrically with respect to $1/2$, the components of the argument of the density $g(u, v)$ will play a symmetric^a role and we shall have $g(u, v) = g(v, u)$, for any $(u, v) \in [0, 1]^2$. For convenience, in the following we shall use the symbol z for either of the variables u or v and \bar{z} for the other. Also, to write shortly the characteristic function of a system of *conditions* (equalities or inequalities in variables u and v), we denote:

$$\varphi(\text{conditions}) = \begin{cases} 1, & \text{if conditions hold true for } (z, \bar{z}) = (u, v) \text{ or } (z, \bar{z}) = (v, u); \\ 0, & \text{else,} \end{cases}$$

and

$$\tilde{\varphi}(\text{conditions}) = \begin{cases} 1, & \text{if conditions hold true for } (z, \bar{z}) = (u, v) \text{ and } (z, \bar{z}) = (v, u); \\ 0, & \text{else.} \end{cases}$$

Theorem 1. *The local density in the unit square of points $(q'/Q, q''/Q)$, where q' and q'' are denominators of neighbor fractions in $\mathfrak{F}_{Q, \text{even}}$, approaches a limiting density g as*

^aWe use the word *symmetric* for tuples with components listed in reverse order of one another, and also for points situated symmetrically with respect to the first diagonal.

$Q \rightarrow \infty$. Moreover, for any real numbers u, v with $0 \leq u, v \leq 1$,

$$\begin{aligned}
g(u, v) = & \sum_{j=1}^{\infty} \frac{1}{j} \tilde{\varphi} \left(z < 1; \ j < \frac{z + \bar{z}}{1 - z} \right) \\
& + \sum_{j=1}^{\infty} \frac{1}{2j} \left\{ \varphi \left((j+1)z + \bar{z} = j, \text{ if } \frac{j-1}{j+1} < z < \frac{j}{j+2} \right) \right. \\
& \quad \left. + \varphi \left(z = 1, \text{ if } \frac{j-1}{j+1} < \bar{z} < 1 \right) \right\} \\
& + \sum_{j=1}^{\infty} \left\{ \frac{2j+1}{8j(j+1)} \varphi \left(z = \frac{j-1}{j+1}; \ \bar{z} = 1 \right) \right. \\
& \quad \left. + \frac{j+2}{4j(j+1)} \tilde{\varphi} \left(z = \frac{j}{j+2} \right) + \frac{1}{4j} \tilde{\varphi} \left(z = 1 \right) \right\}.
\end{aligned} \tag{2}$$

Figures 1-4 show how the unit square is covered by countably many polygons, on the interior of which the local density function $g(u, v)$ is constant. The explicit value of that constant is provided by Theorem 1. We remark that for a point (u, v) , which is an interior point of one of the polygons that form the above covering of the unit square, the second sum and the third sum on the right side of (2) vanish, and in the first sum only finitely many terms are nonzero, namely those corresponding to the values of $j \geq 1$ for which one has simultaneously $j < (u+v)/(1-v)$ and $j < (u+v)/(1-u)$. So, $g(u, v)$ reduces in this case to a partial sum of the harmonic series, with more and more terms of the series to be counted as the point (u, v) is chosen closer and closer to the upper-right corner $(1, 1)$ of the unit square. Additional terms, contained in the second and in the third sum on the right side of (2), only appear in the case when (u, v) lies on one of the sides or coincides with one of the vertices of one of the polygons that form the covering of the unit square.

Turning to the same problem on shorter intervals, if \mathcal{I} does not have $1/2$ as midpoint, than the symmetry $g^{\mathcal{I}}(u, v) = g^{\mathcal{I}}(v, u)$ is apriori not at all obvious. The next theorem provides the stronger result that $g^{\mathcal{I}}(u, v)$ not only exists, for any $\mathcal{I} \subseteq [0, 1]$, but that it is independent of \mathcal{I} .

Theorem 2. *Let $\mathcal{I} \subseteq [0, 1]$ of length > 0 . Then*

$$g^{\mathcal{I}}(u, v) = g(v, u), \quad \text{for any } (u, v) \in [0, 1] \times [0, 1]. \tag{3}$$

Note that, for any $Q \geq 2$, pictures like those from Figures A and B are symmetric, point by point, with respect to the first diagonal, and this also holds in the case of $\mathfrak{F}_{Q, \text{odd}}^{\mathcal{I}}$

and $\mathfrak{F}_{Q,\text{even}}^{\mathcal{I}}$, for any interval \mathcal{I} symmetric with respect to $1/2$. If \mathcal{I} is not centered at $1/2$, then the corresponding pictures are never symmetric point by point, but Theorem 2 guarantees that their limits, as $Q \rightarrow \infty$, are symmetric.

By comparing Theorem 1 above with Theorem 1 of [10], we find that the limiting local density function $g(u, v)$ is the same for both subsequences $\mathfrak{F}_{Q,\text{odd}}$ and $\mathfrak{F}_{Q,\text{even}}$. Therefore, Corollary 1 and Proposition 1 from [10] also hold for $\mathfrak{F}_{Q,\text{even}}$.

Let $\mathcal{D}(0, 2)$ denote the set of limit points of sequences of pairs $((q'_n/Q_n, q''_n/Q_n))_{n \in \mathbf{N}}$, with $Q_n \rightarrow \infty$ and q'_n, q''_n denominators of consecutive fractions in $\mathfrak{F}_{Q,\text{even}}$.

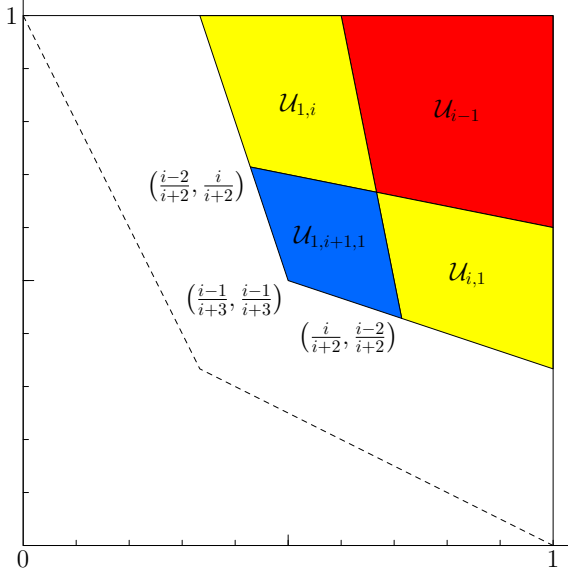
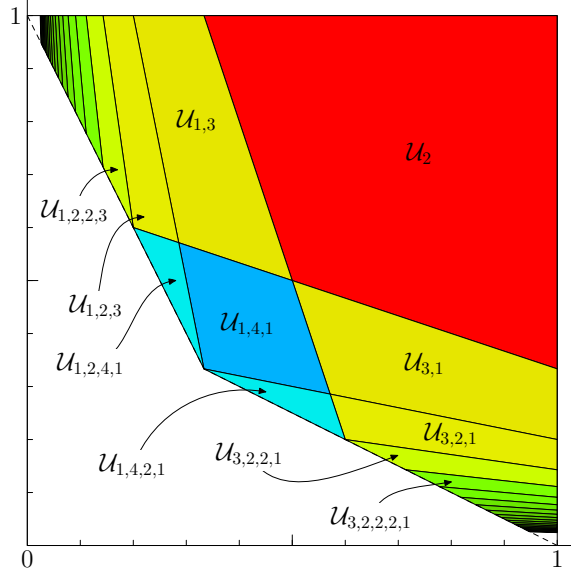
Corollary 1. *The set $\mathcal{D}(0, 2)$ coincides with the quadrilateral bounded by the lines: $y = 1$, $x = 1$, $2x + y = 1$, and $2y + x = 1$.*

We remark that the more uniform distribution of $\mathcal{D}_{500,\text{odd}}$ compared to $\mathcal{D}_{500,\text{even}}$ in Figures A and B is due to a combination of two factors that occur for Q small. The first one is the preponderance in $\mathfrak{F}_{Q,\text{odd}}$ of pairs of type $\mathbf{T}(0)$, that is, of neighbor odd fractions in \mathfrak{F}_Q , and the second is the fact that the cardinality of $\mathfrak{F}_{Q,\text{odd}}$ is about twice as large as the cardinality of $\mathfrak{F}_{Q,\text{even}}$.

The next corollary provides the probability that a neighbor pair of denominators in $\mathfrak{F}_{Q,\text{even}}$ is *small*, in the sense that their sum does not exceed Q .

Corollary 2. *The probability that the sum of neighbor denominators of fractions from $\mathfrak{F}_{Q,\text{even}}$ is $\leq Q$ approaches $1/6$, as $Q \rightarrow \infty$.*

In order to prove Theorem 1, we begin by presenting in Sections 2 and 3 some geometric prerequisites, in particular the tessellation of the Farey triangle, and then continue in Sections 4 and 5 with the study of different types of pairs of neighbor denominators of fractions in $\mathfrak{F}_{Q,\text{even}}$. These conclude with Theorem 3, in which $g_r(u, v)$, the local density at level r , is written as a sum of a series of special quantities assigned to the pieces of the tessellations of the Farey triangle. This result plays a key role in the proof of Theorem 1 from Section 6, where we put together all the pieces. Remarkably, these pieces fit into a large mosaic composed by a series of superimposed puzzles at odd levels (see Figures 5-6) recovering in the even case the same density as that obtained in [10] in the odd case. In the last section we employ a technique which makes use of estimates for Kloosterman sums [13], [22], in order to prove Theorem 2.

Figure 5: The baby puzzle \mathfrak{G}_i .Figure 6: The big puzzle \mathfrak{G} .

2. SOME GEOMETRY OF THE FAREY FRACTIONS

We state here the basic properties of the Farey series needed for the rest of the paper. The first one, already mentioned in the Introduction, says that if $a'/q' < a''/q''$ are neighbor fractions in \mathfrak{F}_Q , then

$$a''q' - a'q'' = 1. \quad (4)$$

Next, suppose that $a'/q' < a''/q'' < a'''/q'''$ are consecutive elements of \mathfrak{F}_Q . Then, the middle fraction, called the *mediant*, is given by

$$\frac{a''}{q''} = \frac{a' + a'''}{q' + q''}. \quad (5)$$

This shows that the mediant fraction is reduced by an integer k , called the *index* of the Farey fraction a'/q' , that satisfies:

$$k = \frac{a' + a'''}{a''} = \frac{q' + q'''}{q''} = a''q' - a'q''' = \left\lceil \frac{Q + q'}{q''} \right\rceil. \quad (6)$$

We state a third important property in the following lemma.

Lemma 1. *The positive integers q', q'' are denominators of neighbor fractions in \mathfrak{F}_Q if and only if $(q', q'') \in \mathcal{T}_Q$ and $\gcd(q', q'') = 1$. Also, the pair (q', q'') appears exactly once as a pair of denominators of consecutive Farey fractions.*

For the proof of relations (4) and (5), observed for the first time in particular cases by Haros and Farey, we refer to Hardy and Wright [18], while for Lemma 1, and further developments, see Hall [14], [15], Hall and Tenenbaum [17], and Hall and Shiu [16].

Now let $(q', q'', q''', \dots, q^{(h)})$ denote a generic h -tuple of denominators of neighbor fractions in \mathfrak{F}_Q . Then, we see that relation (6) can be employed together with Lemma 1 to obtain a characterization of any such h -tuple in terms of its first two components. Moreover, while any pair (q', q'') with coprime components $\leq Q$ does appear exactly once as a pair of neighbor denominators of Farey fractions, the components of longer tuples must satisfy additional conditions in order to appear together as neighbor denominators of fractions in \mathfrak{F}_Q . We write these conditions using the index.

For any positive integer k , we consider the convex polygon defined by

$$\mathcal{T}_{Q,k} := \{(x, y) : 0 < x, y \leq Q, x + y > Q, ky \leq Q + x < (k+1)y\}.$$

These are quadrilaterals, except for $k = 1$, when $\mathcal{T}_{Q,1}$ is a triangle. The vertices of $\mathcal{T}_{Q,1}$ are $(0, Q)$, $(\frac{Q}{3}, \frac{2Q}{3})$, (Q, Q) , and for any $k \geq 2$, the vertices of $\mathcal{T}_{Q,k}$ are $(Q, \frac{2Q}{k})$; $(\frac{Q(k-1)}{k+1}, \frac{2Q}{k+1})$; $(\frac{Qk}{k+2}, \frac{2Q}{k+2})$; $(Q, \frac{2Q}{k+1})$. Scaling by a factor of Q , for any $k \geq 1$ we get $\mathcal{T}_k = \mathcal{T}_{Q,k}/Q$, a bounded polygon inside the unit square, which is independent of Q .

On each \mathcal{T}_k the index function, defined by $(x, y) \mapsto [\frac{1+x}{y}]$, is locally constant. Therefore the polygons \mathcal{T}_k can be used to describe the triplets of neighbor denominators of Farey fractions. We state this as a lemma.

Lemma 2. *The positive integers q', q'', q''' are denominators of consecutive fractions in \mathfrak{F}_Q and $k = \frac{q'+q''}{q''}$ if and only if $(q', q'') \in \mathcal{T}_{Q,k}$ and $\gcd(q', q'') = 1$.*

We remark that the sets \mathcal{T}_k , with $k \geq 1$, are disjoint and they form a partition of \mathcal{T} .

In many instances, one can estimate the number of Farey fractions with a certain property by counting the number of lattice points in a suitable domain. In this respect, the following lemma, which is a variation of Lemma 2 from [1], is very useful. For any domain $\Omega \subset \mathbb{R}^2$ we denote:

$$N_{\text{odd,odd}}(\Omega) := \#\{(x, y) \in \Omega \cap \mathbb{Z}^2 : x \text{ odd}, y \text{ odd}, \gcd(x, y) = 1\},$$

$$N_{\text{even,odd}}(\Omega) := \#\{(x, y) \in \Omega \cap \mathbb{Z}^2 : x \text{ even}, y \text{ odd}, \gcd(x, y) = 1\},$$

$$N_{\text{odd,even}}(\Omega) := \#\{(x, y) \in \Omega \cap \mathbb{Z}^2 : x \text{ odd}, y \text{ even}, \gcd(x, y) = 1\}.$$

These numbers can be estimated by using Möbius summation. One has the following asymptotic formulas.

Lemma 3 ([2], Corollary 3.2). *Let $R_1, R_2 > 0$, and $R \geq \min(R_1, R_2)$. Then, for any region $\Omega \subseteq [0, R_1] \times [0, R_2]$ with rectifiable boundary, we have:*

$$N_{\text{odd,odd}}(\Omega) = 2\text{Area}(\Omega)/\pi^2 + O(C_{R,\Omega}),$$

$$N_{\text{odd,even}}(\Omega) = 2\text{Area}(\Omega)/\pi^2 + O(C_{R,\Omega}),$$

where $C_{R,\Omega} = \text{Area}(\Omega)/R + R + \text{length}(\partial\Omega) \log R$.

Then, for Ω as in Lemma 3, one immediately gets

$$N_{\text{even,odd}}(\Omega) = 2\text{Area}(\Omega)/\pi^2 + O(C_{R,\Omega}).$$

3. THE POLYGONS $\mathcal{T}_{\mathbf{k}}$

We consider the map $T: \mathcal{T} \rightarrow \mathcal{T}$, defined by

$$T(x, y) = \left(y, \left[\frac{1+x}{y} \right] y - x \right),$$

which was introduced and studied in [3]. This transformation is invertible and its inverse is given by

$$T^{-1}(x, y) = \left(\left[\frac{1+y}{x} \right] x - y, x \right).$$

One should notice that if $a'/q' < a''/q'' < a'''/q'''$ are consecutive elements in \mathfrak{F}_Q , then $T(q'/Q, q''/Q) = (q''/Q, q'''/Q)$. Then, for any $\mathbf{k} = (k_1, \dots, k_r) \in (\mathbb{N}^*)^r$, we put

$$\mathcal{T}_{\mathbf{k}} = \mathcal{T}_{k_1} \cap T^{-1}\mathcal{T}_{k_2} \cap \dots \cap T^{-r+1}\mathcal{T}_{k_r}.$$

We use the notational convention of dropping extra parentheses, so for example when $r = 1$ and $k \in \mathbb{N}^*$, we have $\mathcal{T}_{(k)} = \mathcal{T}_k = \{(x, y) \in \mathcal{T} : \left[\frac{1+x}{y} \right] = k\}$. Also, to avoid double subscripts, at small levels r we often write k, l, \dots , rather than k_1, k_2, \dots .

We remark that at any level r , the convex polygons $\mathcal{T}_{\mathbf{k}}$ are pieces of a partition of \mathcal{T} . The structure of these partitions is studied in [12]. Here we only need the polygons assigned to tuples \mathbf{k} , whose components have special parities. These are those tuples \mathbf{k} with all components even, except for the first and the last when $r \geq 2$. We call these tuples *admissible*, and for each level r , we denote by $\mathcal{A}(r)$ the set of admissible r -tuples.

The polygons needed in the sequel are listed in Table 1. Notice that here $\mathcal{T}_{\mathbf{k}}$ is a quadrilateral, except for $\mathbf{k} = (1, 2, 4, 1)$ and $\mathbf{k} = (1, 4, 2, 1)$, when it is a triangle.

Table 1: The polygons $\mathcal{T}_{\mathbf{k}}$, for all admissible \mathbf{k} .

Level	\mathbf{k}	No. of vertices	Vertices of $\mathcal{T}_{\mathbf{k}}$
1	$k \geq 2$, even	4	$(\frac{k-1}{k+1}, \frac{2}{k+1})$; $(\frac{k}{k+2}, \frac{2}{k+2})$; $(1, \frac{2}{k+1})$; $(1, \frac{2}{k})$
2	1, 3	4	$(\frac{1}{5}, \frac{4}{5})$; $(\frac{2}{7}, \frac{5}{7})$; $(\frac{1}{2}, 1)$; $(\frac{1}{3}, 1)$
2	$(1, l)$, with $l \geq 5$ odd	4	$(\frac{l-3}{l+1}, \frac{l-1}{l+1})$; $(\frac{l-2}{l+2}, \frac{l}{l+2})$; $(\frac{l-1}{l+1}, 1)$; $(\frac{l-2}{l}, 1)$
2	3, 1	4	$(\frac{1}{2}, \frac{1}{2})$; $(\frac{4}{7}, \frac{3}{7})$; $(1, \frac{3}{5})$; $(1, \frac{2}{3})$
2	$(k, 1)$, with $k \geq 5$ odd	4	$(\frac{k-1}{k+1}, \frac{2}{k+1})$; $(\frac{k}{k+2}, \frac{2}{k+2})$; $(1, \frac{2}{k+1})$; $(1, \frac{2}{k})$
3	1, 2, 3	4	$(\frac{1}{7}, \frac{6}{7})$; $(\frac{1}{5}, \frac{4}{5})$; $(\frac{2}{7}, 1)$; $(\frac{1}{5}, 1)$
3	3, 2, 1	4	$(\frac{4}{7}, \frac{3}{7})$; $(\frac{3}{5}, \frac{2}{5})$; $(1, \frac{4}{7})$; $(1, \frac{3}{5})$
3	1, 4, 1	4	$(\frac{2}{7}, \frac{5}{7})$; $(\frac{1}{3}, \frac{2}{3})$; $(\frac{4}{7}, 1)$; $(\frac{1}{2}, 1)$
3	$(1, l, 1)$, with $l \geq 6$, even	4	$(\frac{l-3}{l+1}, \frac{l-1}{l+1})$; $(\frac{l-2}{l+2}, \frac{l}{l+2})$; $(\frac{l-1}{l+1}, 1)$; $(\frac{l-2}{l}, 1)$
4	1, 2, 2, 3	4	$(\frac{1}{9}, \frac{8}{9})$; $(\frac{1}{7}, \frac{6}{7})$; $(\frac{1}{5}, 1)$; $(\frac{1}{7}, 1)$
4	3, 2, 2, 1	4	$(\frac{3}{5}, \frac{2}{5})$; $(\frac{5}{7}, \frac{3}{7})$; $(1, \frac{5}{9})$; $(1, \frac{4}{7})$
4	1, 2, 4, 1	3	$(\frac{1}{5}, \frac{4}{5})$; $(\frac{1}{3}, 1)$; $(\frac{2}{7}, 1)$
4	1, 4, 2, 1	3	$(\frac{1}{3}, \frac{2}{3})$; $(\frac{3}{5}, 1)$; $(\frac{4}{7}, 1)$
r	1, 2, \dots , 2, 3	4	$(\frac{1}{2r+1}, \frac{2r}{2r+1})$; $(\frac{1}{2r-1}, \frac{2r-2}{2r-1})$; $(\frac{1}{2r-3}, 1)$; $(\frac{1}{2r-1}, 1)$
r	3, 2, \dots , 2, 1	4	$(\frac{2r-5}{2r-3}, \frac{r-2}{2r-3})$; $(\frac{2r-3}{2r-1}, \frac{r-1}{2r-1})$; $(1, \frac{r+1}{2r+1})$; $(1, \frac{r}{2r-1})$

We shall denote $\mathcal{T}_{Q,\mathbf{k}} := Q\mathcal{T}_{\mathbf{k}}$.

4. THE COMPONENTS OF $\mathcal{D}(0, 2)$

Since any two consecutive denominators of fractions in \mathfrak{F}_Q are coprime, it follows that between any two neighbor fractions in $\mathfrak{F}_{Q,\text{even}}$ there must be at least one odd fraction from \mathfrak{F}_Q . We note that the number of these odd fractions may be quite large as Q increases. We classify the pairs of neighbor even fractions according to the number of odd intermediate fractions. So, we say that the pair (γ', γ'') of consecutive fractions in $\mathfrak{F}_{Q,\text{even}}$ is of *type* $T(r)$ if there exist exactly r odd fractions between γ' and γ'' in \mathfrak{F}_Q . In this case, we shall also say that the pairs (q', q'') and $(q'/Q, q''/Q)$ are of type $T(r)$, where q', q'' are the denominators of γ', γ'' , respectively.

Geometry inside the Farey triangle helps one locate the points corresponding to such pairs. Next, we determine, one by one, the contribution of pairs of each type $T(r)$ to $\mathcal{D}(0, 2)$. We denote by $\mathcal{E}(r)$ the contribution to $\mathcal{D}(0, 2)$ of points of order $T(r)$. Then, we have

$$\mathcal{D}(0, 2) = \bigcup_{r=1}^{\infty} \mathcal{E}(r). \quad (7)$$

In the following we find explicitly each set $\mathcal{E}(r)$.

4.1. Points of type T(1). By (6), it follows that pairs of fractions of this type have as denominators the end points of a triple $(q', q'', kq'' - q')$, with q' even, q'' odd and $k_1 = k = \lceil \frac{Q+q'}{q''} \rceil$ even. This means that, for any even $k \geq 2$, we need to retain the lattice points in the domain

$$\mathcal{U}_{Q,k} = \{(q', kq'' - q') : (q', q'') \in \mathcal{T}_{Q,k}\},$$

with q' even, q'' odd, and $\gcd(q', q'') = 1$. In the limit, when $Q \rightarrow \infty$, these points produce a subset of $\mathcal{D}(0, 2)$ that is dense in the quadrilateral with vertices

$$\mathcal{U}_k = \left\{ \left(\frac{k-1}{k+1}, 1 \right); \left(\frac{k}{k+2}, \frac{k}{k+2} \right); \left(1, \frac{k-1}{k+1} \right); (1, 1) \right\}.$$

Thus, we have

$$\mathcal{E}(1) = \bigcup_{\substack{k=2 \\ k \text{ even}}}^{\infty} \mathcal{U}_k.^a \quad (8)$$

We remark that all \mathcal{U}_k , not only those with k even, contribute to $\mathcal{D}(1, 2)$ (see [10]).

4.2. Points of type T(2). These points come from 4-tuples of parity (e, o, o, e) and this requires both k and l to be odd. Let

$$\mathcal{U}_{Q,k,l} = \{(q', l(kq'' - q') - q'') : (q', q'') \in \mathcal{T}_{Q,k,l}\},$$

and we need to pick up the lattice points in $\mathcal{U}_{Q,k,l}$ where q' even, q'' odd, and $\gcd(q', q'') = 1$. In the limit, when $Q \rightarrow \infty$, these points produce sequences of subsets of $\mathcal{D}(0, 2)$ that are dense in each of the quadrilaterals with vertices

$$\begin{aligned} \mathcal{U}_{1,3} &= \{(1/5, 1); (2/7, 4/7); (1/2, 1/2); (1/3, 1)\}, \\ \mathcal{U}_{3,1} &= \{(1/2, 1/2); (4/7, 2/7); (1, 1/5); (1, 1/3)\}, \\ \mathcal{U}_{1,l} &= \left\{ \left(\frac{l-3}{l+1}, 1 \right); \left(\frac{l-2}{l+2}, \frac{l}{l+2} \right); \left(\frac{l-1}{l+1}, \frac{l-1}{l+1} \right); \left(\frac{l-2}{l}, 1 \right) \right\}, \quad \text{for } l \geq 5, \\ \mathcal{U}_{k,1} &= \left\{ \left(\frac{k-1}{k+1}, \frac{k-1}{k+1} \right); \left(\frac{k}{k+2}, \frac{k-2}{k+2} \right); \left(1, \frac{k-3}{k+1} \right); \left(1, \frac{k-2}{k} \right) \right\}, \quad \text{for } k \geq 5. \end{aligned}$$

Then

$$\mathcal{E}(2) = \mathcal{U}_{1,3} \cup \mathcal{U}_{3,1} \cup \bigcup_{\substack{k=5 \\ k \text{ odd}}}^{\infty} \mathcal{U}_{k,1} \cup \bigcup_{\substack{l=5 \\ l \text{ odd}}}^{\infty} \mathcal{U}_{1,l}. \quad (9)$$

^aMost often, we denote a polygon by its vertices and, depending on the context, the same notation will be used to indicate its topological closure.

4.3. Points of type T(3). These points are produced by 5-tuples of parity (e, o, o, o, e) and this requires both k and m to be odd and l even. Let

$$\mathcal{U}_{Q,k,l,m} = \{ (q', m(l(kq'' - q') - q'') - (kq'' - q')) : (q', q'') \in \mathcal{T}_{Q,k,l,m} \},$$

and we need to pick up the lattice points in $\mathcal{U}_{Q,k,l,m}$ where q' even, q'' odd, and $\gcd(q', q'') = 1$. In the limit, when $Q \rightarrow \infty$, these points produce the sequence of subsets of $\mathcal{D}(0, 2)$ that are dense in each of the quadrilaterals with vertices

$$\mathcal{U}_{1,2,3} = \{ (1/7, 1); (1/5, 3/5); (2/7, 4/7); (1/5, 1) \},$$

$$\mathcal{U}_{3,2,1} = \{ (4/7, 2/7); (3/5, 1/5); (1, 1/7); (1, 1/5) \},$$

$$\mathcal{U}_{1,4,1} = \{ (2/7, 4/7); (1/3, 1/3); (4/7, 2/7); (1/2, 1/2) \},$$

$$\mathcal{U}_{1,l,1} = \left\{ \left(\frac{l-3}{l+1}, \frac{l-1}{l+1} \right); \left(\frac{l-2}{l+2}, \frac{l-2}{l+2} \right); \left(\frac{l-1}{l+1}, \frac{l-3}{l+1} \right); \left(\frac{l-2}{l}, \frac{l-2}{l} \right) \right\}, \quad \text{for } l \geq 6.$$

Then

$$\mathcal{E}(3) = \mathcal{U}_{1,2,3} \cup \mathcal{U}_{3,2,1} \cup \bigcup_{\substack{l=4 \\ l \text{ even}}}^{\infty} \mathcal{U}_{1,l,1}. \quad (10)$$

4.4. Points of type T(4). These points are produced by 6-tuples of parity (e, o, o, o, o, e) and this requires both k, n to be odd and m, l even. Let

$$\mathcal{U}_{Q,k,l,m,n} = \{ (q', n(m(l(kq'' - q') - q'') - (kq'' - q')) - (l(kq'' - q') - q'')) : (q', q'') \in \mathcal{T}_{Q,k,l,m,n} \}.$$

As before, we need to pick up the lattice points in $\mathcal{U}_{Q,k,l,m,n}$ with q' even, q'' odd, and $\gcd(q', q'') = 1$. By Table 1, we know that there are only four such sets. In the limit, as $Q \rightarrow \infty$, we obtain two quadrilaterals and two triangles:

$$\mathcal{U}_{1,2,2,3} = \{ (1/9, 1); (1/7, 5/7); (1/5, 3/5); (1/7, 1) \},$$

$$\mathcal{U}_{3,2,2,1} = \{ (3/5, 1/5); (5/7, 1/7); (1, 1/9); (1, 1/7) \},$$

$$\mathcal{U}_{1,2,4,1} = \{ (1/5, 3/5); (1/3, 1/3); (2/7, 4/7) \},$$

$$\mathcal{U}_{1,4,2,1} = \{ (1/3, 1/3); (3/5, 1/5); (4/7, 2/7) \}.$$

Then

$$\mathcal{E}(4) = \mathcal{U}_{1,2,2,3} \cup \mathcal{U}_{3,2,2,1} \cup \mathcal{U}_{1,2,4,1} \cup \mathcal{U}_{1,4,2,1}. \quad (11)$$

4.5. **Points of type $T(r)$, $r \geq 5$.** These points are produced by the $(r+2)$ -tuples $(q', \dots, q^{(r+2)})$ of parity (e, o, \dots, o, e) . It turns out that the only possible corresponding r -tuples \mathbf{k} are $(1, 2, \dots, 2, 3)$ and its symmetric $(3, 2, \dots, 2, 1)$, with $(r-2)$ 2's between the endpoints. This allows us to find a closed formula for $q^{(r+2)}$. We find that $q^{(r+2)} = -(2r-1)q' + 2q''$ in the first case, and $q^{(r+2)} = -q' + 2q''$ in the second case. Thus, we define

$$\mathcal{U}_{Q,1,2,\dots,2,3} = \left\{ (q', -(2r-1)q' + 2q'') : (q', q'') \in \mathcal{T}_{Q,1,2,\dots,2,3} \right\},$$

and

$$\mathcal{U}_{Q,3,2,\dots,2,1} = \left\{ (q', -q' + 2q'') : (q', q'') \in \mathcal{T}_{Q,3,2,\dots,2,1} \right\}.$$

As before, only the lattice points with q' even, q'' odd, and $\gcd(q', q'') = 1$ should be considered, and in the limit, when $Q \rightarrow \infty$, for any given r , we obtain the quadrilaterals:

$$\begin{aligned} \mathcal{U}_{1,2,\dots,2,3} &= \left\{ \left(\frac{1}{2r+1}, 1 \right); \left(\frac{1}{2r-1}, \frac{2r-3}{2r-1} \right); \left(\frac{1}{2r-3}, \frac{2r-5}{2r-3} \right); \left(\frac{1}{2r-1}, 1 \right) \right\}, \quad \text{for } r \geq 5, \\ \mathcal{U}_{3,2,\dots,2,1} &= \left\{ \left(\frac{2r-5}{2r-3}, \frac{1}{2r-3} \right); \left(\frac{2r-3}{2r-1}, \frac{1}{2r-1} \right); \left(1, \frac{1}{2r+1} \right); \left(1, \frac{1}{2r-1} \right) \right\}, \quad \text{for } r \geq 5. \end{aligned}$$

Then

$$\mathcal{E}(r) = \mathcal{U}_{1,2,\dots,2,3} \cup \mathcal{U}_{3,2,\dots,2,1}, \quad \text{for } r \geq 5. \quad (12)$$

In Figure 3, one can see a representation of $\mathcal{D}(0, 2)$ covered by \mathcal{U}_* . In addition, the union from the right hand side of (7) gives a first hint on the local densities on $\mathcal{D}(0, 2)$. Complete calculations are postponed to Section 5.

5. THE DENSITY OF POINTS OF TYPE $T(r)$

Let $g(x, y)$ be the function that gives the local density of points $(q'/Q, q''/Q)$ in the unit square as $Q \rightarrow \infty$, and let $g_r(x, y)$ be the local density in the unit square of the points $(q'/Q, q''/Q)$ of type $T(r)$, as $Q \rightarrow \infty$. At any point $(u, v) \in (0, 1)^2$, this local density is defined by

$$g_r(u, v) := \lim_{\text{Area}(\square) \rightarrow 0} \frac{\lim_{Q \rightarrow \infty} \frac{\#(\square \cap \mathcal{D}_Q(0, 2)/Q)}{\#\mathcal{D}_Q(0, 2)}}{\text{Area}(\square)}, \quad (13)$$

where $\square \subset \mathbb{R}^2$ are squares centered at (u, v) . They are related by

$$g(u, v) = \sum_{r=1}^{\infty} g_r(u, v), \quad (14)$$

provided we show that each local density $g_r(u, v)$ exists as $Q \rightarrow \infty$. In the following we find each $g_r(u, v)$. The proof generalizes that of Theorem 1 from [10, Section 3.2].

5.1. Generalities on g_r . Let $r \geq 1$, let (x_0, y_0) be a fixed point inside the unit square $(0, 1)^2$, and fix a small $\eta > 0$. In the following, the superscript \mathcal{L} indicates the last element of a tuple. For example, $q^{\mathcal{L}} = q_r^{\mathcal{L}}(q', q'')$ is the denominator of the $(r + 2)$ -nd fraction in \mathfrak{F}_Q , starting with q', q'' . We denote the square centered at (x_0, y_0) by $\square = \square_\eta(x_0, y_0) = (x_0 - \eta, x_0 + \eta) \times (y_0 - \eta, y_0 + \eta)$.

By definition, any pair $(q', q^{\mathcal{L}})$ of type $\mathbf{T}(r)$ is generated by an $(r+2)$ -tuple $(q', q'', \dots, q^{\mathcal{L}})$ of denominators of consecutive fractions in \mathfrak{F}_Q , with $q', q^{\mathcal{L}}$ even, and the rest of the components odd. We consider the set \mathcal{B}_Q of pairs (q', q'') of type $\mathbf{T}(r)$ for which the corresponding point $(q'/Q, q^{\mathcal{L}}/Q)$ falls in \square , that is,

$$\mathcal{B}_Q(r) = \left\{ (q', q'') \in \mathbb{N}^2 : \begin{array}{l} 1 \leq q', q'' \leq Q, \gcd(q', q'') = 1, q' + q'' > Q, \\ q' \text{ even, } q'' \text{ odd; } \mathbf{k}(q', q'') \in \mathcal{A}(r), (q', q^{\mathcal{L}}(r)) \in Q \cdot \square \end{array} \right\}.$$

The cardinality of \mathcal{B}_Q is $\#\mathcal{B}_Q = N_{\text{even, odd}}(\Omega_Q)$, where $\Omega_Q = \Omega_Q(r) = \Omega_Q(x_0, y_0, \eta)(r)$ is given by

$$\Omega_Q(r) = \left\{ (x, y) \in \mathbb{R}^2 : \begin{array}{l} 1 \leq x, y \leq Q, x + y > Q, \\ \mathbf{k}(x, y) \in \mathcal{A}(r), (x, x_r^{\mathcal{L}}(x, y)) \in Q \cdot \square \end{array} \right\}.$$

Here $x^{\mathcal{L}} = x_r^{\mathcal{L}}(x, y)$ is the $(r + 2)$ -nd element of the sequence defined recursively by: $x_{-1} = x$, $x_0 = y$ and $x_j = k_j x_{j-1} - x_{j-2}$, where $k_j = \lfloor \frac{1+x_{j-2}}{x_{j-1}} \rfloor$, for $1 \leq j \leq r$ and $\mathbf{k}(x, y) = (k_1, \dots, k_r) \in \mathcal{A}(r)$.

Multiplying by $1/Q$, we obtain the bounded set

$$\Omega(r) = \left\{ (x, y) \in (0, 1)^2 : x + y > 1, \mathbf{k}(x, y) \in \mathcal{A}(r), (x, x_r^{\mathcal{L}}(x, y)) \in \square \right\},$$

and $Q \cdot \Omega(r) = \Omega_Q(r)$. We are interested in the area of $\Omega(r)$, since, by Lemma 3, we know that

$$\#\mathcal{B}_Q(r) = \frac{2Q^2 \text{Area}(\Omega(r))}{\pi^2} + O(Q \log Q). \quad (15)$$

Next we split $\Omega(r)$ into the pieces given by the shadows left on it by each of the sets

$$\mathcal{U}_{\mathbf{k}}(r) := \left\{ (x, y) \in (0, 1)^2 : \mathbf{k}_r(x, y) = \mathbf{k} \right\}, \quad \text{for } \mathbf{k} \in \mathcal{A}(r).$$

Since $\Omega(r) \subset \mathcal{T}$ and $\mathcal{U}_{\mathbf{k}}(r) \cap \mathcal{T} = \mathcal{T}_{\mathbf{k}}$, it follows that $\Omega(r) \cap \mathcal{U}_{\mathbf{k}}(r) = \mathcal{T}_{\mathbf{k}} \cap \mathcal{P}_{\mathbf{k}}(r)$, where $\mathcal{P}_{\mathbf{k}}(r) = \mathcal{P}_{\mathbf{k},r}(x_0, y_0, \eta)$ is the parallelogram

$$\mathcal{P}_{\mathbf{k}}(r) = \{(x, y) \in \mathbb{R}^2 : (x, x_r^{\mathcal{L}}(x, y)) \in \square_{\eta}(x_0, y_0)\}.$$

Thus, we have obtained

$$\text{Area}(\Omega(r)) = \sum_{\mathbf{k} \in \mathcal{A}(r)} \text{Area}(\mathcal{T}_{\mathbf{k}} \cap \mathcal{P}_{\mathbf{k}}(r)). \quad (16)$$

A compactness argument shows that, although $\mathcal{A}(r)$ may be infinite, only finitely many terms of the series are non-zero. Our next objective is to make explicit their size in terms of the position of (x_0, y_0) . But first we need to establish a concrete expression for $\mathcal{P}_{\mathbf{k}}(r)$.

5.2. The index $p_r(\mathbf{k})$ and the parallelogram $\mathcal{P}_{\mathbf{k}}(r)$. First, we define the sequence of polynomials $p_r(\mathbf{k})$ by: $p_0(\cdot) = 1$, $p_1(k_1) = k_1$, and then recursively, for any $r \geq 2$,

$$p_r(k_1, \dots, k_r) = k_r p_{r-1}(k_1, \dots, k_{r-1}) - p_{r-2}(k_1, \dots, k_{r-2}). \quad (17)$$

The first polynomials are:

$$p_2(\mathbf{k}) = k_1 k_2 - 1;$$

$$p_3(\mathbf{k}) = k_1 k_2 k_3 - k_1 - k_3;$$

$$p_4(\mathbf{k}) = k_1 k_2 k_3 k_4 - k_1 k_2 - k_1 k_4 - k_3 k_4 + 1;$$

$$p_5(\mathbf{k}) = k_1 k_2 k_3 k_4 k_5 - k_1 k_2 k_3 - k_1 k_2 k_5 - k_1 k_4 k_5 - k_3 k_4 k_5 + k_1 + k_3 + k_5.$$

Some other particular values we need later are:

$$\begin{aligned} p_r(1, 2, \dots, 2) &= 1, & \text{for } r \geq 2, \\ p_r(1, 2, \dots, 2, 3) &= 2, & \text{for } r \geq 2, \\ p_r(2, \dots, 2, 3) &= 2r + 1, & \text{for } r \geq 2, \end{aligned} \quad (18)$$

Also, we remark the symmetry property:

$$p_r(k_r, \dots, k_1) = p_r(k_1, \dots, k_r). \quad (19)$$

More on this fundamental sequence of polynomials can be found in [12].

Next, let us notice that $x_r^{\mathcal{L}}(x, y)$ is a linear combination of x and y :

$$x_r^{\mathcal{L}}(x, y) = p_r(k_1, \dots, k_r)y - p_{r-1}(k_2, \dots, k_r)x. \quad (20)$$

Returning now to our parallelogram, by (20) we see that this is the set of points $(x, y) \in \mathbb{R}^2$ that satisfy the conditions:

$$\begin{cases} x_0 - \eta < x < x_0 + \eta, \\ y_0 - \eta < p_r(k_1, \dots, k_r)y - p_{r-1}(k_2, \dots, k_r)x < y_0 + \eta. \end{cases}$$

From this, we see that the area of $\mathcal{P}_{\mathbf{k}}(r)$ is

$$\text{Area}(\mathcal{P}_{\mathbf{k}}(r)) = \frac{4\eta^2}{p_r(\mathbf{k})}, \quad (21)$$

and its center has coordinates

$$C_{\mathbf{k}}(r) = \left(x_0, \frac{p_{r-1}(k_2, \dots, k_r)}{p_r(k_1, \dots, k_r)}x_0 + \frac{1}{p_r(k_1, \dots, k_r)}y_0 \right). \quad (22)$$

5.3. The density $g_r(u, v)$ II. In order to obtain a concrete expression for the density, one requires an explicit form of the series in (16). Since $\eta > 0$ can be chosen as small as we please, it follows that the summands there depend on the position of (x_0, y_0) with respect to $\mathcal{T}_{\mathbf{k}}$. In the following, we shall assume that η is small enough. We may also assume that \mathbf{k} is bounded, since all the parallelograms $\mathcal{P}_{\mathbf{k}}(r)$ are contained in the vertical strip given by the inequalities $x_0 - \eta < x < x_0 + \eta$, and since only finitely many polygons $\mathcal{T}_{\mathbf{k}}$ intersect this strip.

Let now \mathbf{k} be an admissible r -tuple. We check what happens when $C_{\mathbf{k}}(r)$ lies on the interior $\overset{\circ}{\mathcal{T}}_{\mathbf{k}}$ of $\mathcal{T}_{\mathbf{k}}$, on the edges of $\mathcal{T}_{\mathbf{k}}$, or in the set $V(\mathbf{k})$ of vertices of $\mathcal{T}_{\mathbf{k}}$.

Firstly, if $C_{\mathbf{k}}(r) \in \overset{\circ}{\mathcal{T}}_{\mathbf{k}}$, it follows that $\mathcal{P}_{\mathbf{k}}(r) \subset \mathcal{T}_{\mathbf{k}}$, so $\text{Area}(\mathcal{T}_{\mathbf{k}} \cap \mathcal{P}_{\mathbf{k}}(r)) = \text{Area}(\mathcal{P}_{\mathbf{k}}(r))$. Secondly, if $C_{\mathbf{k}}(r) \in \partial\mathcal{T}_{\mathbf{k}} \setminus V(\mathcal{T}_{\mathbf{k}})$, then $\text{Area}(\mathcal{T}_{\mathbf{k}} \cap \mathcal{P}_{\mathbf{k}}(r)) = \text{Area}(\mathcal{P}_{\mathbf{k}}(r))/2$, since any line that crosses $\mathcal{P}_{\mathbf{k}}(r)$ through its center cuts the parallelogram into two pieces of equal area. Thirdly, suppose $C_{\mathbf{k}}(r) \in V(\mathcal{T}_{\mathbf{k}})$. Then $\text{Area}(\mathcal{T}_{\mathbf{k}} \cap \mathcal{P}_{\mathbf{k}}(r))$ depends on the angle formed by the corresponding edges, and these angles may differ for different vertices of $\mathcal{T}_{\mathbf{k}}$ or for different values of \mathbf{k} . We have collected all the results in Table 2. In the calculations, we have made use of the relations (18), (19) and (20). For each $V \in V(\mathcal{T}_{\mathbf{k}})$ and $C_{\mathbf{k}}(r) = V$, for a concise presentation, we have translated $\mathcal{T}_{\mathbf{k}} \cap \mathcal{P}_{\mathbf{k}}(r)$ with a vector V to the origin. Also, we remark that the size of $\mathcal{T}_{\mathbf{k}} \cap \mathcal{P}_{\mathbf{k}}(r)$ is always proportional to η . It follows that $\alpha_{\mathbf{k}}(V) := \text{Area}(\mathcal{T}_{\mathbf{k}} \cap \mathcal{P}_{\mathbf{k}}(r))/\eta^2$ is independent of η .

The nice thing about this calculation is that it has an error-correcting check. This is due to another remarkable property of the parallelogram, which implies that the entries

in the column of $\alpha_{\mathbf{k}}(V)$ satisfy the relations

$$\sum_V \alpha_{\mathbf{k}}(V) = \begin{cases} \frac{\text{Area}(\mathcal{P}_{\mathbf{k}}(r))}{\eta^2} = \frac{4}{p_r(\mathbf{k})}, & \text{if } \mathcal{T}_{\mathbf{k}} \text{ is a quadrilateral;} \\ \frac{\text{Area}(\mathcal{P}_{\mathbf{k}}(r))}{2\eta^2} = \frac{2}{p_r(\mathbf{k})}, & \text{if } \mathcal{T}_{\mathbf{k}} \text{ is a triangle,} \end{cases} \quad (23)$$

for each admissible \mathbf{k} . Here the summation is over all the vertices of $\mathcal{T}_{\mathbf{k}}$. For example, if $\mathbf{k} = (1, l, 1)$, with $l \geq 6$ even, we have

$$\frac{2l-1}{2(l-1)l} + \frac{l+2}{(l-2)l} + \frac{2l-1}{2(l-1)l} + \frac{l}{(l-2)(l-1)} = \frac{4}{l-2}$$

and for $\mathbf{k} = (1, 2, 4, 1)$, we have

$$\frac{7}{24} + \frac{3}{40} + \frac{19}{30} = 1.$$

Table 2: The vertices and the area of the polygons $\mathcal{T}_{\mathbf{k}} \cap P_{\mathbf{k}}(r)$ for $C_{\mathbf{k}}(r)$ equal to each of the vertices of $\mathcal{T}_{\mathbf{k}}$. Each polygon was scaled by $1/\eta$ and translated into the origin.

\mathbf{k}	Vertex $V = C_{\mathbf{k}}$	Vertices of $\frac{1}{\eta}(\mathcal{T}_{\mathbf{k}} \cap P_{\mathbf{k}}(r)) - C_{\mathbf{k}}(r)$	$\alpha_{\mathbf{k}}(V)$
$k \geq 2$, even	$(\frac{k-1}{k+1}, \frac{2}{k+1})$	$(0, 0); (\frac{1}{k+1}, -\frac{1}{k+1}); (1, 0); (1, \frac{1}{k})$	$\frac{2k+1}{2k(k+1)}$
	$(\frac{k}{k+2}, \frac{2}{k+2})$	$(0, 0); (1, \frac{1}{k+1}); (1, \frac{2}{k}); (-\frac{1}{k+1}, \frac{1}{k+1})$	$\frac{k+2}{k(k+1)}$
	$(1, \frac{2}{k+1})$	$(0, 0); (0, \frac{1}{k}); (-1, 0); (-1, -\frac{1}{k+1})$	$\frac{2k+1}{2k(k+1)}$
	$(1, \frac{2}{k})$	$(0, 0); (-1, -\frac{1}{k}); (-1, -\frac{2}{k}); (0, -\frac{1}{k})$	$\frac{1}{k}$
1, 3	$(\frac{1}{5}, \frac{4}{5})$	$(0, 0); (\frac{1}{5}, -\frac{1}{5}); (1, 1); (1, \frac{3}{2})$	$\frac{9}{20}$
	$(\frac{2}{7}, \frac{5}{7})$	$(0, 0); (1, \frac{4}{3}); (1, 2); (-\frac{1}{5}, \frac{1}{5})$	$\frac{19}{30}$
	$(\frac{1}{2}, 1)$	$(0, 0); (-\frac{1}{3}, 0); (-1, -1); (-1, -\frac{4}{3})$	$\frac{1}{3}$
	$(\frac{1}{3}, 1)$	$(0, 0); (-1, -\frac{3}{2}); (-1, -2); (\frac{1}{3}, 0)$	$\frac{7}{12}$
$(1, l)$, with $l \geq 5$ odd	$(\frac{l-3}{l+1}, \frac{l-1}{l+1})$	$(0, 0); (\frac{2}{l+1}, \frac{1}{l+1}); (1, 1); (1, \frac{l}{l-1})$	$\frac{l}{(l-1)(l+1)}$
	$(\frac{l-2}{l+2}, \frac{l}{l+2})$	$(0, 0); (1, \frac{l+1}{l}); (1, \frac{l+1}{l-1}); (-\frac{2}{l+1}, -\frac{1}{l+1})$	$\frac{2l^2+5l+1}{2(l-1)l(l+1)}$
	$(\frac{l-1}{l+1}, 1)$	$(0, 0); (-\frac{1}{l}, 0); (-1, -1); (-1, -\frac{l+1}{l})$	$\frac{1}{l}$
	$(\frac{l-2}{l}, 1)$	$(0, 0); (-1, -\frac{l}{l-1}); (-1, -\frac{l+1}{l-1}); (\frac{1}{l}, 0)$	$\frac{2l+1}{2l(l-1)}$
3, 1	$(\frac{1}{2}, \frac{1}{2})$	$(0, 0); (\frac{1}{3}, -\frac{1}{3}); (1, 0); (1, \frac{1}{3})$	$\frac{1}{3}$
	$(\frac{4}{7}, \frac{3}{7})$	$(0, 0); (1, \frac{2}{5}); (1, 1); (-\frac{1}{3}, \frac{1}{3})$	$\frac{19}{30}$
	$(1, \frac{3}{5})$	$(0, 0); (0, \frac{1}{2}); (-1, 0); (-1, -\frac{2}{5})$	$\frac{9}{20}$
	$(1, \frac{2}{3})$	$(0, 0); (-1, -\frac{1}{3}); (-1, -1); (0, -\frac{1}{2})$	$\frac{7}{12}$
$(k, 1)$, with $k \geq 5$ odd	$(\frac{k-1}{k+1}, \frac{2}{k+1})$	$(0, 0); (\frac{1}{k}, -\frac{1}{k}); (1, 0); (1, \frac{1}{k})$	$\frac{1}{k}$
	$(\frac{k}{k+2}, \frac{2}{k+2})$	$(0, 0); (1, \frac{1}{k+1}); (1, \frac{2}{k-1}); (-\frac{1}{k}, \frac{1}{k})$	$\frac{2k^2+5k+1}{2(k-1)k(k+1)}$
	$(1, \frac{2}{k+1})$	$(0, 0); (0, \frac{1}{k-1}); (-1, 0); (-1, -\frac{1}{k+1})$	$\frac{k}{(k-1)(k+1)}$
	$(1, \frac{2}{k})$	$(0, 0); (-1, -\frac{1}{k}); (-1, -\frac{2}{k-1}); (0, -\frac{1}{k-1})$	$\frac{2k+1}{2k(k+1)}$
1, 2, 3	$(\frac{1}{7}, \frac{6}{7})$	$(0, 0); (\frac{1}{7}, -\frac{1}{7}); (1, 2); (1, \frac{5}{2})$	$\frac{13}{28}$
	$(\frac{1}{5}, \frac{4}{5})$	$(0, 0); (1, \frac{7}{3}); (1, 3); (-\frac{1}{7}, \frac{1}{7})$	$\frac{13}{21}$

continued on next page

continued from previous page

\mathbf{k}	Vertex $V = C_{\mathbf{k}}(r)$	Vertices of $\frac{1}{\eta}(\mathcal{T}_{\mathbf{k}} \cap P_{\mathbf{k}}(r)) - C_{\mathbf{k}}(r)$	$\alpha_{\mathbf{k}}(V)$
	$(\frac{2}{7}, 1)$	$(0, 0); (-\frac{1}{5}, 0); (-1, -2); (-1, -\frac{7}{3})$	$\frac{11}{30}$
	$(\frac{1}{5}, 1)$	$(0, 0); (-1, -\frac{5}{2}); (-1, -3); (\frac{1}{5}, 0)$	$\frac{11}{20}$
3, 2, 1	$(\frac{4}{7}, \frac{3}{7})$	$(0, 0); (\frac{1}{3}, -\frac{1}{3}); (1, 0); (1, \frac{2}{5})$	$\frac{11}{30}$
	$(\frac{3}{5}, \frac{2}{5})$	$(0, 0); (1, \frac{3}{7}); (1, 1); (-\frac{1}{3}, \frac{1}{3})$	$\frac{13}{21}$
	$(1, \frac{4}{7})$	$(0, 0); (0, \frac{1}{2}); (-1, 0); (-1, -\frac{3}{7})$	$\frac{13}{28}$
	$(1, \frac{3}{5})$	$(0, 0); (-1, -\frac{2}{5}); (-1, -1); (0, -\frac{1}{2})$	$\frac{11}{20}$
1, 4, 1	$(\frac{2}{7}, \frac{5}{7})$	$(0, 0); (\frac{1}{5}, -\frac{1}{5}); (1, 1); (1, \frac{4}{3})$	$\frac{11}{30}$
	$(\frac{1}{3}, \frac{2}{3})$	$(0, 0); (1, \frac{7}{5}); (1, 2); (-\frac{1}{5}, \frac{1}{5})$	$\frac{3}{5}$
	$(\frac{4}{7}, 1)$	$(0, 0); (-\frac{1}{3}, 0); (-1, -1); (-1, -\frac{7}{5})$	$\frac{11}{30}$
	$(\frac{1}{2}, 1)$	$(0, 0); (-1, -\frac{4}{3}); (-1, -2); (\frac{1}{3}, 0)$	$\frac{2}{3}$
$(1, l, 1)$, with $l \geq 6$ even	$(\frac{l-3}{l+1}, \frac{l-1}{l+1})$	$(0, 0); (\frac{2}{l}, \frac{1}{l}); (1, 1); (1, \frac{l}{l-1})$	$\frac{2l-1}{2(l-1)l}$
	$(\frac{l-2}{l+2}, \frac{l}{l+2})$	$(0, 0); (1, \frac{l+1}{l}); (1, \frac{l}{l-2}); (-\frac{2}{l}, -\frac{1}{l})$	$\frac{l+2}{(l-2)l}$
	$(\frac{l-1}{l+1}, 1)$	$(0, 0); (-\frac{1}{l-1}, 0); (-1, -1); (-1, -\frac{l+1}{l})$	$\frac{2l-1}{2(l-1)l}$
	$(\frac{l-2}{l}, 1)$	$(0, 0); (-1, -\frac{l}{l-1}); (-1, -\frac{l}{l-2}); (\frac{1}{l-1}, 0)$	$\frac{l}{(l-2)(l-1)}$
1, 2, 2, 3	$(\frac{1}{9}, \frac{8}{9})$	$(0, 0); (\frac{1}{9}, -\frac{1}{9}); (1, 3); (1, \frac{7}{2})$	$\frac{17}{36}$
	$(\frac{1}{7}, \frac{6}{7})$	$(0, 0); (\frac{1}{2}, \frac{5}{4}); (1, 3); (-\frac{1}{9}, \frac{1}{9})$	$\frac{65}{72}$
	$(\frac{1}{5}, 1)$	$(0, 0); (-\frac{1}{7}, 0); (-\frac{1}{2}, -\frac{5}{4})$	$\frac{5}{56}$
	$(\frac{1}{7}, 1)$	$(0, 0); (-1, -\frac{7}{2}); (-1, -4); (\frac{1}{7}, 0)$	$\frac{15}{28}$
1, 2, 4, 1	$(\frac{1}{5}, \frac{4}{5})$	$(0, 0); (\frac{1}{2}, \frac{3}{4}); (1, 2); (1, \frac{7}{3})$	$\frac{7}{24}$
	$(\frac{1}{3}, 1)$	$(0, 0); (-\frac{1}{5}, 0); (-\frac{1}{2}, -\frac{3}{4})$	$\frac{3}{40}$
	$(\frac{2}{7}, 1)$	$(0, 0); (-1, -\frac{7}{3}); (-1, -3); (\frac{1}{5}, 0)$	$\frac{19}{30}$
1, 4, 2, 1	$(\frac{1}{3}, \frac{2}{3})$	$(0, 0); (1, \frac{5}{4}); (1, \frac{7}{5})$	$\frac{3}{40}$
	$(\frac{3}{5}, 1)$	$(0, 0); (-\frac{1}{3}, 0); (-1, -1); (-1, -\frac{5}{4})$	$\frac{7}{24}$
	$(\frac{4}{7}, 1)$	$(0, 0); (-1, -\frac{7}{5}); (-1, -2); (\frac{1}{3}, 0)$	$\frac{19}{30}$
3, 2, 2, 1	$(\frac{3}{5}, \frac{2}{5})$	$(0, 0); (1, \frac{1}{4}); (1, \frac{3}{7})$	$\frac{5}{56}$
	$(\frac{5}{7}, \frac{3}{7})$	$(0, 0); (1, \frac{4}{9}); (1, 1); (-1, 0); (-1, -\frac{1}{4})$	$\frac{65}{72}$
	$(1, \frac{5}{9})$	$(0, 0); (0, \frac{1}{2}); (-1, 0); (-1, -\frac{4}{9})$	$\frac{17}{36}$
	$(1, \frac{4}{7})$	$(0, 0); (-1, -\frac{3}{7}); (-1, -1); (0, -\frac{1}{2})$	$\frac{15}{28}$
1, 2, ..., 2, 3	$(\frac{1}{2r+1}, \frac{2r}{2r+1})$	$(0, 0); (\frac{1}{2r+1}, -\frac{1}{2r+1}); (1, r-1); (1, \frac{2r-1}{2})$	$\frac{4r+1}{4(2r+1)}$
	$(\frac{1}{2r-1}, \frac{2(r-1)}{2r-1})$	$(0, 0); (\frac{1}{2}, \frac{2r-3}{4}); (1, r-1); (1, r); (-\frac{1}{2r+1}, \frac{1}{2r+1})$	$\frac{14r+9}{8(2r+1)}$
	$(\frac{1}{2r-3}, 1)$	$(0, 0); (-\frac{1}{2r-1}, 0); (-\frac{1}{2}, -\frac{2r-3}{4})$	$\frac{2r-3}{8(2r-1)}$
	$(\frac{1}{2r-1}, 1)$	$(0, 0); (-1, -\frac{2r-1}{2}); (-1, -r); (\frac{1}{2r-1}, 0)$	$\frac{4r-1}{4(2r-1)}$
3, 2, ..., 2, 1	$(\frac{2r-5}{2r-3}, \frac{r-2}{2r-3})$	$(0, 0); (1, \frac{1}{4}); (1, \frac{r-1}{2r-1})$	$\frac{2r-3}{8(2r-1)}$
	$(\frac{2r-3}{2r-1}, \frac{r-1}{2r-1})$	$(0, 0); (1, \frac{r}{2r+1}); (1, 1); (-1, 0); (-1, -\frac{1}{4})$	$\frac{14r+9}{8(2r+1)}$
	$(1, \frac{r+1}{2r+1})$	$(0, 0); (0, \frac{1}{2}); (-1, 0); (-1, -\frac{r}{2r+1})$	$\frac{4r+1}{4(2r+1)}$
	$(1, \frac{r}{2r-1})$	$(0, 0); (-1, -\frac{r-1}{2r-1}); (-1, -1); (0, -\frac{1}{2})$	$\frac{4r-1}{4(2r-1)}$

These observations combined with (21) put (16) into the form

$$\text{Area}(\Omega(r)) = 4\eta^2 \sum_{C_{\mathbf{k}}(r) \in \mathring{\mathcal{T}}_{\mathbf{k}}} \frac{1}{p_r(\mathbf{k})} + 2\eta^2 \sum_{C_{\mathbf{k}}(r) \in \partial\mathcal{T}_{\mathbf{k}} \setminus V(\mathcal{T}_{\mathbf{k}})} \frac{1}{p_r(\mathbf{k})} + \eta^2 \sum_{C_{\mathbf{k}}(r) \in V(\mathcal{T}_{\mathbf{k}})} \alpha_{\mathbf{k}}(r). \quad (24)$$

We now immediately obtain a corresponding expression for g_r . An application of Lemma 3 (see also Lemma 4 below) provides

$$\#\mathfrak{F}_{Q,\text{even}} = Q^2/\pi^2 + O(Q \log Q).$$

Using (15) and the fact that the number of points $(q'/Q, q''/Q)$ from $(0, 1)^2$, where q' and q'' are denominators of two consecutive elements from $\mathfrak{F}_{Q,\text{even}}$ is $\#\mathfrak{F}_{Q,\text{even}} - 1$, we have:

$$\iint_{\square_{\eta}(x_0, y_0)} g_r(x, y) dx dy = \lim_{Q \rightarrow \infty} \frac{\#\mathcal{B}_Q(r)}{\#\mathfrak{F}_{Q,\text{even}} - 1} = 2\text{Area}(\Omega(r)). \quad (25)$$

Then, by the Lebesgue differentiation theorem, we have $g_r(x_0, y_0) = \lim_{\eta \rightarrow 0} \text{Area}(\Omega(r))/(4\eta^2)$, which combined with (24) gives the following result.

Theorem 3. *For $(x_0, y_0) \in [0, 1]^2$ and any integer $r \geq 1$, we have:*

$$g_r(x_0, y_0) = \sum_{C_{\mathbf{k}}(r) \in \mathring{\mathcal{T}}_{\mathbf{k}}} \frac{2}{p_r(\mathbf{k})} + \sum_{C_{\mathbf{k}}(r) \in \partial\mathcal{T}_{\mathbf{k}} \setminus V(\mathcal{T}_{\mathbf{k}})} \frac{1}{p_r(\mathbf{k})} + \frac{1}{2} \sum_{C_{\mathbf{k}}(r) \in V(\mathcal{T}_{\mathbf{k}})} \alpha_{\mathbf{k}}(r), \quad (26)$$

where the sums run over r -tuples \mathbf{k} which are admissible.

We call the generic term in the first sum the *kernel* of \mathbf{k} . Notice that if $C_{\mathbf{k}}(r)$ is on the boundary the terms added in the second sum are equal to half of the kernel, and in the third sum the kernel distributes in sizes proportional to the angles of different vertices of the polygon $\mathcal{T}_{\mathbf{k}}$.

6. PROOF OF THEOREM 1

Our starting point in the proof of Theorem 1 is formula (26), in which we need to make explicit the conditions of summation in terms of the variables. Since some of the polygons with all components of \mathbf{k} small do not follow the general pattern, at this point we can not get an explicit closed formula for the density, even if we restrict to pairs of a given order. Consequently, we further split the sums on the right hand side of (26). For this, we collect in $h_1^u(x_0, y_0)$ the sum of the terms from the right hand side of (26) with $r = 1$ and $\mathbf{k} = k \geq 4$, even. Similarly, for $r = 2$, $h_2^u(x_0, y_0)$ will be the sum over $\mathbf{k} = (1, l)$ and $\mathbf{k} = (k, 1)$ with $k, l \geq 5$, both odd, while for $r = 3$, $h_3^u(x_0, y_0)$ will be the

sum over $\mathbf{k} = (1, l, 1)$, with $l \geq 6$, even. The remaining terms will be collected separately in $h_1^d(x_0, y_0)$, $h_2^d(x_0, y_0)$ and $h_3^d(x_0, y_0)$. Thus, we have:

$$g_j(x_0, y_0) = h_j^d(x_0, y_0) + h_j^u(x_0, y_0), \quad \text{for } j = 1, 2, 3.$$

Next, we treat one by one each of these terms. We shall assume everywhere, unless otherwise specified, that $x_0, y_0 > 0$. To shorten the notation, we shall use the variables z_0 and \bar{z}_0 for either x_0 or y_0 , with the meaning explained in the Introduction for z, \bar{z} and u, v . In the same way, we extend the notation for the characteristic functions $\varphi, \tilde{\varphi}$ for conditions expressed in terms of z_0, \bar{z}_0 .

6.1. The density $h_1^u(x_0, y_0)$. In this case $r = 1$ and $\mathbf{k} = k \geq 4$ is even, although the first part of the calculation holds true more generally. Then, by (22), we know that $C_k := C_{\mathbf{k}}(1) = (x_0, \frac{x_0+y_0}{k})$, and by the definition of $\mathcal{T}_{\mathbf{k}}$ we know that $C_k \in \overset{\circ}{\mathcal{T}}_k$ if and only if the following conditions hold simultaneously:

$$\begin{cases} 1 - \frac{x_0+y_0}{k} < x_0 < 1, \\ \frac{x_0+1}{k+1} < \frac{x_0+y_0}{k} < \frac{x_0+1}{k}. \end{cases}$$

Since we assumed that $x_0, y_0 > 0$, this translates into the equivalence

$$C_k \in \overset{\circ}{\mathcal{T}}_k \iff 0 < x_0, y_0 < 1 \text{ and } k < \frac{x_0 + y_0}{1 - \min(x_0, y_0)}, \quad \text{for } k \geq 2.$$

For the edges of \mathcal{T}_k , we obtain

$$C_k(1) \in \partial \mathcal{T}_k \setminus V(\mathcal{T}_k) \iff \begin{cases} k = \frac{x_0+y_0}{1-x_0}, & \text{for } \frac{k-1}{k+1} < x_0 < \frac{k}{k+2}; \\ \text{or} \\ k = \frac{x_0+y_0}{1-y_0}, & \text{for } \frac{k}{k+2} < x_0 < 1; \\ \text{or} \\ x_0 = 1, & \text{for } \frac{k-1}{k+1} < y_0 < 1; \\ \text{or} \\ y_0 = 1, & \text{for } \frac{k-1}{k+1} < x_0 < 1, \end{cases}$$

and for the vertices of \mathcal{T}_k , we have:

$$C_k \in V(\mathcal{T}_k) \iff (x_0, y_0) \in \left\{ \left(\frac{k-1}{k+1}, 1 \right); \left(\frac{k}{k+2}, \frac{k}{k+2} \right); \left(1, \frac{k-1}{k+1} \right); (1, 1) \right\} = \mathcal{U}_k.$$

Then, using Table 2, for h_1^u , the corresponding sum from the right-hand side of (26) gives

$$\begin{aligned}
h_1^u(x_0, y_0) = & \sum_{\substack{k \geq 4 \\ k \text{ even}}} \frac{2}{k} \tilde{\varphi}\left(k < \frac{x_0 + y_0}{1 - z_0}; 0 < x_0, y_0 < 1\right) \\
& + \sum_{\substack{k \geq 4 \\ k \text{ even}}} \frac{1}{k} \left\{ \varphi\left(k = \frac{x_0 + y_0}{1 - z_0}, \text{ if } \frac{k-1}{k+1} < z_0 < \frac{k}{k+2}\right) \right. \\
& \quad \left. + \varphi\left(z_0 = 1 \text{ if } \frac{k-1}{k+1} < \bar{z}_0 < 1\right) \right\} \\
& + \sum_{\substack{k \geq 4 \\ k \text{ even}}} \left\{ \frac{2k+1}{4k(k+1)} \varphi\left(z_0 = \frac{k-1}{k+1}; \bar{z}_0 = 1\right) \right. \\
& \quad \left. + \frac{k+2}{2k(k+1)} \tilde{\varphi}\left(z_0 = \frac{k}{k+2}\right) + \frac{2}{k} \tilde{\varphi}\left(z_0 = 1\right) \right\}.
\end{aligned} \tag{27}$$

Here, for a given (x_0, y_0) , in each sum the number of nonzero terms is finite, at most equal to one in the last two sums that correspond to points on the border of \mathcal{U}_k .

6.2. The density $h_2^u(x_0, y_0)$. Here $r = 2$, $\mathbf{k} = (1, l)$ or $\mathbf{k} = (k, 1)$, with $k, l \geq 5$, both odd. We assume first that $\mathbf{k} = (1, l)$. The center of $\mathcal{P}_{1,l}(2)$ has coordinates $C_{1,l} := C_{1,l}(2) = (x_0, \frac{lx_0+y_0}{l-1})$, by (22). Then $C_{1,l} \in \overset{\circ}{\mathcal{T}}_{1,l}$ if and only if the following conditions hold simultaneously:

$$\left\{ \begin{array}{l} \frac{(l-1)\frac{lx_0+y_0}{l-1}-1}{l} < x_0 < \frac{l\cdot\frac{lx_0+y_0}{l-1}-1}{l+1}, \\ \frac{x_0+1}{2} < \frac{lx_0+y_0}{l-1} < 1. \end{array} \right.$$

This gives

$$C_{1,l} \in \overset{\circ}{\mathcal{T}}_{1,l} \iff \left\{ \begin{array}{l} x_0 < l, \ y_0 < 1, \\ \frac{1+y_0}{1-x_0} < l < \min\left(\frac{x_0+y_0+1}{1-x_0}, \frac{x_0+1}{1-y_0}\right), \end{array} \right. \quad \text{for } l \geq 3.$$

Next, the conditions for the open edges of $\mathcal{T}_{1,l}$ are

$$C_{1,l} \in \partial\mathcal{T}_{1,l} \setminus V(\mathcal{T}_{1,l}) \iff \begin{cases} (l+1)x_0 + 2y_0 = l-1, & \text{for } \frac{l-3}{l+1} < x_0 < \frac{l-2}{l+2}; \\ \text{or} \\ x_0 + ly_0 = l-1, & \text{for } \frac{l-2}{l+2} < x_0 < \frac{l-1}{l+1}; \\ \text{or} \\ lx_0 + y_0 = l-1, & \text{for } \frac{l-2}{l} < x_0 < \frac{l-1}{l+1}; \\ \text{or} \\ y_0 = 1, & \text{for } \frac{l-3}{l+1} < x_0 < \frac{l-2}{l}. \end{cases}$$

For the vertices of $\mathcal{T}_{1,l}$, we have

$$C_{1,l} \in V(\mathcal{T}_{1,l}) \iff (x_0, y_0) \in \left\{ \left(\frac{l-3}{l+1}, 1 \right); \left(\frac{l-2}{l+2}, \frac{l}{l+2} \right); \left(\frac{l-1}{l+1}, \frac{l-1}{l+1} \right); \left(\frac{l-2}{l}, 1 \right) \right\} = \mathcal{U}_{1,l}.$$

The symmetry allows us to use the z_0, \bar{z}_0 notation to collect the contribution of terms corresponding to $(1, l)$ and $(k, 1)$ in the same formula. Then, using Table 2, the corresponding sum from the right-hand side of (26) gives

$$\begin{aligned} h_2^u(x_0, y_0) = & \sum_{\substack{l \geq 5 \\ k \text{ odd}}} \frac{2}{l-1} \varphi \left(z_0, \bar{z}_0 < 1; \frac{1+z_0}{1-\bar{z}_0} < l < \min \left(\frac{2z_0 + \bar{z}_0 + 1}{1-\bar{z}_0}, \frac{\bar{z}_0 + 1}{1-z_0} \right) \right) \\ & + \sum_{\substack{l \geq 5 \\ l \text{ odd}}} \frac{1}{(l-1)} \left\{ \varphi \left((l+1)z_0 + 2\bar{z}_0 = l-1, \text{ if } \frac{l-3}{l+1} < z_0 < \frac{l-2}{l+2} \right) \right. \\ & \quad \left. + \varphi \left(z_0 + l\bar{z}_0 = l-1, \text{ if } \frac{l-2}{l+2} < \bar{z}_0 < 1 \right) \right\} \\ & \quad + \varphi \left(z_0 = 1, \text{ if } \frac{l-3}{l+1} < \bar{z}_0 < \frac{l-2}{l} \right) \Big\} \\ & + \sum_{\substack{l \geq 5 \\ l \text{ odd}}} \left\{ \frac{l}{2(l-1)(l+1)} \varphi \left(z_0 = \frac{l-3}{l+1}; \bar{z}_0 = 1 \right) \right. \\ & \quad + \frac{2l^2 + 5l + 1}{4(l-1)l(l-1)} \varphi \left(z_0 = \frac{l-2}{l+2}; \bar{z}_0 = \frac{l}{l+2} \right) \\ & \quad \left. + \frac{1}{l} \tilde{\varphi} \left(z_0 = \frac{l-1}{l+1} \right) + \frac{2l+1}{4l(l-1)} \varphi \left(z_0 = \frac{l-2}{l}; \bar{z}_0 = 1 \right) \right\}. \end{aligned} \tag{28}$$

Here, for a given (x_0, y_0) , in each sum the number of nonzero terms is finite, at most equal to one in the last two sums that correspond to points on the border of $\mathcal{U}_{1,l,1}$.

6.3. The density $h_3^u(x_0, y_0)$. Now $r = 3$ and $\mathbf{k} = (1, l, 1)$, with $l \geq 6$ even. The center of $\mathcal{P}_{1,l,1}(3)$ is $C_{1,l,1} := C_{1,l,1}(3) = (x_0, \frac{(l-1)x_0+y_0}{l-1})$, cf. (22), and $\mathcal{T}_{1,l,1} = \mathcal{T}_{1,l}$. Then $C_{1,l,1} \in \mathring{\mathcal{T}}_{1,l,1}$ if and only if the following conditions hold simultaneously:

$$\left\{ \begin{array}{l} \frac{(l-1)\frac{(l-1)x_0+y_0}{l-2}-1}{l} < x_0 < \frac{l\frac{(l-1)x_0+y_0}{l-2}-1}{l+1}, \\ \frac{x_0+1}{2} < \frac{(l-1)x_0+y_0}{l-2} < 1. \end{array} \right.$$

This can be rewritten as

$$C_{1,l,1} \in \mathring{\mathcal{T}}_{1,l,1} \iff \left\{ \begin{array}{l} x_0 - y_0 + 2 < l(1 - y_0), \\ l(1 - y_0) < 2(1 + x_0), \\ l(1 - x_0) < 2(y_0 + 1), \\ y_0 - x_0 + 2 < l(1 - x_0). \end{array} \right.$$

The conditions for the open edges of $\mathcal{T}_{1,l,1}$ are

$$C_{1,l,1} \in \partial\mathcal{T}_{1,l,1} \setminus V(\mathcal{T}_{1,l,1}) \iff \left\{ \begin{array}{l} lx_0 + 2y_0 = l - 2, \quad \text{for } \frac{l-3}{l+1} < x_0 < \frac{l-2}{l+2}; \\ \text{or} \\ 2x_0 + ly_0 = l - 2, \quad \text{for } \frac{l-2}{l+2} < x_0 < \frac{l-1}{l+1}; \\ \text{or} \\ (l-1)x_0 + y_0 = l - 2, \quad \text{for } \frac{l-2}{l} < x_0 < \frac{l-1}{l+1}; \\ \text{or} \\ x_0 + (l-1)y_0 = l - 2, \quad \text{for } \frac{l-3}{l+1} < x_0 < \frac{l-2}{l}. \end{array} \right.$$

For the vertices of $\mathcal{T}_{1,l,1}$, we have:

$$C_{1,l,1} \in V(\mathcal{T}_{1,l,1}) \iff (x_0, y_0) \in \left\{ \left(\frac{l-3}{l+1}, \frac{l-1}{l+1} \right); \left(\frac{l-2}{l+2}, \frac{l-2}{l+2} \right); \left(\frac{l-1}{l+1}, \frac{l-3}{l+1} \right); \left(\frac{l-2}{l}, \frac{l-2}{l} \right) \right\} = \mathcal{U}_{1,l,1}.$$

Then, using Table 2, the corresponding sum from the right-hand side of (26) gives

$$\begin{aligned}
h_3^u(x_0, y_0) = & \sum_{\substack{l \geq 6 \\ k \text{ even}}} \frac{2}{l-2} \tilde{\varphi}\left(z_0 - \bar{z}_0 + 2 < l(1 - \bar{z}_0); \ l(1 - z_0) < 2(\bar{z}_0 + 1)\right) \\
& + \sum_{\substack{l \geq 6 \\ l \text{ even}}} \frac{1}{(l-2)} \left\{ \varphi\left(lz_0 + 2\bar{z}_0 = l-2, \text{ if } \frac{l-2}{l+2} < z_0 < \frac{l-3}{l+1}\right) \right. \\
& \quad \left. + \varphi\left(z_0 + (l-1)\bar{z}_0 = l-2, \text{ if } \frac{l-3}{l+1} < \bar{z}_0 < \frac{l-2}{l}\right) \right\} \\
& + \sum_{\substack{l \geq 6 \\ l \text{ even}}} \left\{ \frac{2l-1}{4l(l-1)} \varphi\left(z_0 = \frac{l-3}{l+1}; \ \bar{z}_0 = \frac{l-1}{l+1}\right) \right. \\
& \quad \left. + \frac{l+2}{2l(l-2)} \tilde{\varphi}\left(z_0 = \frac{l-2}{l+2}\right) + \frac{l}{2(l-2)(l-1)} \tilde{\varphi}\left(z_0 = \frac{l-2}{l}\right) \right\}.
\end{aligned} \tag{29}$$

Here, for a given (x_0, y_0) , in each sum the number of nonzero terms is finite, at most equal to one in the last two sums that correspond to points on the border of $\mathcal{U}_{1,l,1}$.

6.4. The contribution to the density of points of type $T(r)$, $r \geq 5$. In this section we find the tail density, which we define to be

$$g^u(x_0, y_0) = \sum_{r \geq 5} g_r(x_0, y_0).$$

For any $r \geq 5$, there exist only two admissible r -tuples \mathbf{k} , symmetric to one another. Let $\mathbf{k} = (1, 2, \dots, 2, 3)$ be such an r -tuple. Then, since $p_r(\mathbf{k}) = 2$ and $p_{r-1}(2, \dots, 3) = 2r-1$, relation (22) produces $C_{\mathbf{k}} := C_{\mathbf{k}}(3) = (x_0, \frac{(2r-1)x_0+y_0}{2})$. Then $C_{\mathbf{k}} \in \overset{\circ}{T}_{\mathbf{k}}$ if and only if the following conditions hold simultaneously:

$$\begin{cases} \frac{2 \frac{(2r-1)x_0+y_0}{2} - 1}{2r-1} < x_0 < \frac{2 \frac{(2r-1)x_0+y_0}{2} - 1}{2r-3}, \\ 1 - x_0 < \frac{(2r-1)x_0+y_0}{2} < 1. \end{cases}$$

This can be rewritten as

$$C_{\mathbf{k}} \in \overset{\circ}{T}_{\mathbf{k}} \iff \begin{cases} y_0 < 1, \\ 1 < 2x_0 + y_0, \\ 2 < (2r+1)x_0 + y_0, \\ 2 > (2r-1)x_0 + y_0. \end{cases}$$

The conditions for the open edges of $\mathcal{T}_{\mathbf{k}}$ are

$$C_{\mathbf{k}} \in \partial\mathcal{T}_{\mathbf{k}} \setminus V(\mathcal{T}_{\mathbf{k}}) \iff \begin{cases} (2r+1)x_0 + y_0 = 2, & \text{for } \frac{1}{2r+1} < x_0 < \frac{2r-1}{l+2}; \\ \text{or} \\ 2x_0 + y_0 = 1, & \text{for } \frac{1}{2r-1} < x_0 < \frac{1}{2r-3}; \\ \text{or} \\ (2r-1)x_0 + y_0 = 2, & \text{for } \frac{1}{2r-1} < x_0 < \frac{1}{2r-3}; \\ \text{or} \\ y = 1, & \text{for } \frac{1}{2r+1} < x_0 < \frac{1}{2r-1}. \end{cases}$$

For the vertices of $\mathcal{T}_{\mathbf{k}}$, we have

$$C_{\mathbf{k}} \in V(\mathcal{T}_{\mathbf{k}}) \iff (x_0, y_0) \in \left\{ \left(\frac{1}{2r+1}, 1 \right); \left(\frac{1}{2r-1}, \frac{2r-3}{2r-1} \right); \left(\frac{1}{2r-3}, \frac{2r-5}{2r-3} \right); \left(\frac{1}{2r-1}, 1 \right) \right\} = \mathcal{U}_{\mathbf{k}}.$$

We use the notation with the variables z_0, \bar{z}_0 and the symmetry to write in the same formula the contribution to $g(x_0, y_0)$ of all the terms corresponding to the r -tuples $(1, 2, \dots, 2, 3)$ and $(3, 2, \dots, 2, 1)$, for $r \geq 5$. Using the information from Table 2, we obtain

$$\begin{aligned} g^u(x_0, y_0) = & \sum_{r \geq 5} \varphi(\bar{z}_0 < 1 < 2z_0 + \bar{z}_0; (2r-1)z_0 + \bar{z}_0 < 2 < (2r+1)z_0 + \bar{z}_0) \\ & + \sum_{r \geq 5} \frac{1}{2} \left\{ \varphi\left((2r+1)z_0 + \bar{z}_0 = 2, \text{ if } \frac{1}{2r+1} < z_0 < \frac{1}{2r-1}\right) \right. \\ & \quad + \varphi\left(2z_0 + \bar{z}_0 = 1, \text{ if } \frac{1}{2r-1} < z_0 < \frac{1}{2r-3}\right) \\ & \quad + \varphi\left((2r-1)z_0 + \bar{z}_0 = 2, \text{ if } \frac{1}{2r-1} < z_0 < \frac{1}{2r-3}\right) \\ & \quad \left. + \varphi\left(\bar{z}_0 = 1, \text{ if } \frac{1}{2r+1} < z_0 < \frac{1}{2r-1}\right) \right\} \quad (30) \\ & + \sum_{r \geq 5} \left\{ \frac{4r+1}{8(2r+1)} \varphi\left(z_0 = \frac{1}{2r+1}; \bar{z}_0 = 1\right) \right. \\ & \quad + \frac{14r+9}{16(2r+1)} \varphi\left(z_0 = \frac{1}{2r-1}; \bar{z}_0 = \frac{2r-3}{2r-1}\right) \\ & \quad + \frac{2r-3}{16(2r-1)} \varphi\left(z_0 = \frac{1}{2r-3}; \bar{z}_0 = \frac{2r-5}{2r-3}\right) \\ & \quad \left. + \frac{4r-1}{8(2r-1)} \varphi\left(z_0 = \frac{1}{2r-1}; \bar{z}_0 = 1\right) \right\}. \end{aligned}$$

Here, in each sum, for a given (x_0, y_0) , at most one term is nonzero.

6.5. The baby puzzle. Let $i \geq 5$ be a positive integer. Then, we observe that the polygons $\mathcal{U}_{i-1}, \mathcal{U}_{1,i}, \mathcal{U}_{i,1}, \mathcal{U}_{1,i,1}$ fit perfectly into the quadrilateral with vertices

$$\mathfrak{G}_i = \left\{ \left(\frac{i-3}{i+1}, 1 \right); \left(\frac{i-1}{i+3}, \frac{i-1}{i+3} \right); \left(1, \frac{i-3}{i+1} \right); (1, 1) \right\},$$

(see Figure 5). Another nice aspect of this matching is due to the fact that $p_1(i-1) = p_2(1, i) = p_2(i, 1) = p_3(1, i+1, 1)$, so \mathfrak{G}_i may be viewed as region with constant density at level i , say. Let us notice that $\mathfrak{G}_5 \supset \mathfrak{G}_7 \supset \mathfrak{G}_9 \supset \dots$, so $g(u, v)$ is not locally constant on \mathfrak{G}_i , but the support of $g(u, v)$ is a superposition of quadrilateral steps of constant density (provided we show that a similar property holds for the remaining polygons $\mathcal{U}_{\mathbf{k}}$). Putting together (27), (28), (29), we obtain the next result.

Proposition 1. *For any $(x_0, y_0) \in [0, 1]^2$, we have*

$$\begin{aligned} h^u(x_0, y_0) &:= h_1^u(x_0, y_0) + h_2^u(x_0, y_0) + h_3^u(x_0, y_0) = \\ &= \sum_{\substack{i \geq 5 \\ i \text{ odd}}} \frac{2}{i-1} \tilde{\varphi} \left(z_0 < 1; i < \frac{z_0 + 2\bar{z}_0 + 1}{1 - z_0} \right) \\ &+ \sum_{\substack{i \geq 5 \\ i \text{ odd}}} \frac{1}{(i-1)} \left\{ \varphi \left((i+1)z_0 + 2\bar{z}_0 = i-1, \text{ if } \frac{i-3}{i+1} < z_0 < \frac{i-1}{i+3} \right) \right. \\ &\quad \left. + \varphi \left(z_0 = 1, \text{ if } \frac{i-3}{i+1} < \bar{z}_0 < 1 \right) \right\} \quad (31) \\ &+ \sum_{\substack{i \geq 5 \\ i \text{ odd}}} \left\{ \frac{i}{2(i-1)(i+1)} \varphi \left(z_0 = \frac{i-3}{i+1}; \bar{z}_0 = 1 \right) \right. \\ &\quad \left. + \frac{i+3}{2(i-1)(i+1)} \tilde{\varphi} \left(z_0 = \frac{i-1}{i+3} \right) + \frac{1}{2(i-1)} \tilde{\varphi} \left(z_0 = 1 \right) \right\}. \end{aligned}$$

Proof. We only need to check the equality at the matching corners. We have:

$$\begin{aligned} h^u \left(\frac{i-2}{i+2}, \frac{i}{i+2} \right) &= \frac{1}{2} \left(\alpha_{1,i} \left(\frac{i-2}{i+2}, \frac{i}{i+2} \right) + \alpha_{1,i+1,1} \left(\frac{i-2}{i+2}, \frac{i}{i+2} \right) \right) \\ &= \frac{1}{2} \left(\frac{2i^2 + 5i + 1}{2(i-1)i(i+1)} + \frac{2i+1}{2i(i+1)} \right) = \frac{1}{i-1}, \end{aligned}$$

and

$$\begin{aligned} h^u\left(\frac{i-2}{i}, 1\right) &= \frac{1}{2}\left(\alpha_{1,i}\left(\frac{i-2}{i}, 1\right) + \alpha_{i-1}\left(\frac{i-2}{i}, \frac{2}{i}\right)\right) \\ &= \frac{1}{2}\left(\frac{2i+1}{2i(i-1)} + \frac{2i-1}{2i(i-1)}\right) = \frac{1}{i-1}, \end{aligned}$$

equal each to half of the interior density, since they are on the open edges of \mathfrak{G}_i . By symmetry, we have the same result at $(i/(i+2), (i-2)/(i+2))$ and at $(1, (i-2)/i)$. At the interior matching point, we have

$$\begin{aligned} h^u\left(\frac{i-1}{i+1}, \frac{i-1}{i+1}\right) &= \frac{1}{2}\left(\alpha_{i-1}\left(\frac{i-1}{i+2}, \frac{2}{i+1}\right) + \alpha_{1,i}\left(\frac{i-1}{i+1}, 1\right) + \alpha_{i,1}\left(\frac{i-1}{i+1}, \frac{2}{i+1}\right) + \alpha_{1,i+1,1}\left(\frac{i-1}{i+1}, 1\right)\right) \\ &= \frac{1}{2}\left(\frac{i+1}{i(i-1)} + \frac{2}{i} + \frac{i+1}{i(i-1)}\right) = \frac{2}{i-1}, \end{aligned}$$

which concludes the proof of the proposition. \square

6.6. The big puzzle. We group the terms of lower orders into

$$g^d(x_0, y_0) = h^d(x_0, y_0) + h^u(x_0, y_0) + g_4(x_0, y_0),$$

where $h^d(x_0, y_0) = h_1^d(x_0, y_0) + h_2^d(x_0, y_0) + h_3^d(x_0, y_0)$. It remains to find $h^d(x_0, y_0) + g_4(x_0, y_0)$, as $h^u(x_0, y_0)$ was the object of Section 6.5. In fact, the calculations are special cases of those already performed in sections 6.1-6.4. Here, we shall show that the sum of h^d , g_4 and g^u can be combined into a simpler formula, similar to that of h^u .

Let $\mathcal{M} = \{(2); (1, 3); (3, 1); (1, 2, 3); (3, 2, 1); (1, 4, 1); (1, 2, 4, 1); (1, 4, 2, 1); (1, 2, 2, 3); (3, 2, 2, 1); (1, 2, 2, 2, 3); (3, 2, 2, 2, 1); \dots\}$. The key point in the matching that occurs among the supports of different components of $h^d + g_4 + g^u$ is the fact that the kernel is constant, equal to 1, for any \mathbf{k} contributing to the sum. This follows by the equality $p_r(\mathbf{k}) = 2$, for any $\mathbf{k} \in \mathcal{M}$.

Let \mathfrak{G} be the quadrilateral with vertices $(0, 1); (1/3, 1/3); (1, 0); (1, 1)$. It turns out that \mathfrak{G} is the support of $h^d + g_4 + g^u$, and it looks like a mosaic (that is, no superpositions over interior points occur) composed by all polygons $\mathcal{U}_{\mathbf{k}}$, with $\mathbf{k} \in \mathcal{M}$. Perfect matchings occur (see Figure 6) getting particular cases of the general relations, as follows. (In the nonsymmetric cases we give the statement only for the polygons situated above the first diagonal.) The polygons $U_{1,2,2,3}$ and $U_{1,2,3} \cup \mathcal{U}_{1,2,4,1}$ are given also by the formula for $\mathcal{U}_{1,2,\dots,2,3}$, with $r = 4$ and $r = 3$, respectively. More significantly, on the one hand $\tilde{\mathcal{U}}_{1,4,1} = \mathcal{U}_{1,2,4,1} \cup \mathcal{U}_{1,4,1} \cup \mathcal{U}_{1,4,2,1}$ is given by the formula for $\mathcal{U}_{1,l,1}$, with $l = 4$ and, on the

other hand, $\tilde{\mathcal{U}}_{1,3} = \mathcal{U}_{1,3} \cup \mathcal{U}_{1,2,3} \cup \mathcal{U}_{1,2,2,3} \cup \mathcal{U}_{1,2,2,2,3} \cdots$ is given by the formula for $\mathcal{U}_{1,l}$, with $l = 3$. Then \mathfrak{G} is composed by $\tilde{\mathcal{U}}_{1,3}$, $\tilde{\mathcal{U}}_{1,4,1}$, $\tilde{\mathcal{U}}_{3,1}$, and \mathcal{U}_1 in the same way as $\mathcal{U}_{1,i}$, $\mathcal{U}_{1,i+1,1}$, $\tilde{\mathcal{U}}_{i,1}$, and \mathcal{U}_{i-1} completed the baby puzzle, \mathfrak{G}_i . Next we summarize the contribution to $g(x_0, y_0)$ of all tuples $\mathbf{k} \in \mathcal{M}$.

Proposition 2. *For any $(x_0, y_0) \in [0, 1]^2$, we have*

$$\begin{aligned} g^{\text{du}}(x_0, y_0) &:= h^{\text{d}}(x_0, y_0) + g_4(x_0, y_0) + g^{\text{u}}(x_0, y_0) = \\ &= \tilde{\varphi}(z_0 < 1; 1 < 2z_0 + \bar{z}_0) + \\ &+ \frac{1}{2}\varphi(2z_0 + \bar{z}_0 = 1, \text{ if } 0 < z_0 < 1/3) + \frac{1}{2}\varphi(z_0 = 1, \text{ if } 0 < z_0 < 1) \\ &+ \frac{3}{16}\varphi(z_0 = 0; \bar{z}_0 = 1) + \frac{3}{8}\tilde{\varphi}(z_0 = 1/3) + \frac{1}{4}\tilde{\varphi}(z_0 = 1). \end{aligned} \quad (32)$$

Proof. We just check the equality at the matching vertices using entries from Table 2. Beginning with the border of \mathfrak{G} , for points on the top edge, we have:

$$\begin{aligned} g^{\text{du}}(1/3, 1) &= \frac{1}{2}(\alpha_2(1/3, 2/3) + \alpha_{13}(1/3, 1)) = \frac{1}{2}\left(\frac{5}{12} + \frac{7}{12}\right) = \frac{1}{2}; \\ g^{\text{du}}(1/5, 1) &= \frac{1}{2}(\alpha_{13}(1/5, 4/5) + \alpha_{123}(1/5, 1)) = \frac{1}{2}\left(\frac{9}{20} + \frac{11}{20}\right) = \frac{1}{2}; \\ g^{\text{du}}(1/7, 1) &= \frac{1}{2}(\alpha_{123}(1/7, 6/7) + \alpha_{1223}(1/7, 1)) = \frac{1}{2}\left(\frac{13}{28} + \frac{15}{28}\right) = \frac{1}{2}; \\ g^{\text{du}}\left(\frac{1}{2r+1}, 1\right) &= \frac{1}{2}\left(\alpha_{12\dots 23}\left(\frac{1}{2r+1}, \frac{2r}{2r+1}\right) + \alpha_{122\dots 23}\left(\frac{1}{2r+1}, 1\right)\right) \\ &= \frac{1}{2}\left(\frac{4r+1}{4(2r+1)} + \frac{4(r+1)-1}{4(2(r+1)-1)}\right) = \frac{1}{2}, \quad \text{for } r \geq 4. \end{aligned}$$

On the edge with endpoints $(0, 1)$, $(1/3, 1/3)$, we have:

$$\begin{aligned} g^{\text{du}}(1/5, 3/5) &= \frac{1}{2}(\alpha_{123}(1/5, 4/5) + \alpha_{1223}(1/5, 1) + \alpha_{1241}(1/5, 4/5)) \\ &= \frac{1}{2}\left(\frac{13}{21} + \frac{5}{56} + \frac{7}{24}\right) = \frac{1}{2}; \\ g^{\text{du}}(1/7, 5/7) &= \frac{1}{2}(\alpha_{1223}(1/7, 6/7) + \alpha_{12223}(1/7, 1)) = \frac{1}{2}\left(\frac{65}{72} + \frac{7}{72}\right) = \frac{1}{2}; \\ g^{\text{du}}\left(\frac{1}{2r-1}, \frac{2r-3}{2r-1}\right) &= \frac{1}{2}\left(\alpha_{12\dots 23}\left(\frac{1}{2r-1}, \frac{2(r-1)}{2r-1}\right) + \alpha_{122\dots 23}\left(\frac{1}{2r-1}, 1\right)\right) \\ &= \frac{1}{2}\left(\frac{14r+9}{8(2r+1)} + \frac{2(r+1)-3}{8(2(r+1)-1)}\right) = \frac{1}{2}, \quad \text{for } r \geq 5. \end{aligned}$$

These are all equal to half of the kernel, as expected. The same results hold for points symmetric with respect to the first diagonal.

At the vertices of \mathfrak{G} , we have:

$$\begin{aligned} g^{\text{du}}(1, 1) &= \frac{1}{2}\alpha_2(1, 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}; \\ g^{\text{du}}(1/3, 1/3) &= \frac{1}{2}(\alpha_{141}(1/3, 2/3) + \alpha_{1241}(1/3, 1) + \alpha_{1421}(1/3, 2/3)) \\ &= \frac{1}{2}\left(\frac{3}{5} + \frac{3}{40} + \frac{3}{40}\right) = \frac{3}{18}. \end{aligned}$$

At the corners $(0, 1)$ and $(1, 0)$ the problem is a little bit more complicated. Here, as the inside limit in (13) is taken over Q , the parallelogram $\mathcal{P}_{\mathbf{k}}(r)$ will cover completely $\mathcal{T}_{\mathbf{k}}$, for infinitely many \mathbf{k} from the sequences with 2's embraced by 1 and 3, while some of these $\mathcal{T}_{\mathbf{k}}$'s will be covered partially. Though, we have a quick shot solution to the problem of finding $g^{\text{du}}(0, 1)$ and $g^{\text{du}}(1, 0)$ using, on a larger scale, the property of the parallelogram used in (23) for the quadrilateral with vertices: $(0, 1)$; $(1/3, 1/3)$; $(1/2, 1/2)$; $(1/3, 1)$ and the fact that in \mathfrak{G} the index has everywhere the same value 2. We obtain

$$\begin{aligned} g(0, 1) &= \frac{1}{2}\left(2 - \frac{1}{2}\alpha_{141}(1/2, 1) - \alpha_{13}(1/2, 1) - \alpha_{13}(1/3, 1) \right. \\ &\quad \left. - \alpha_{1241}(1/3, 1) - \frac{1}{2}\alpha_{141}(1/3, 2/3)\right) \\ &= \frac{1}{2}\left(2 - \frac{1}{2} \cdot \frac{2}{3} - \frac{1}{3} - \frac{7}{12} - \frac{3}{40} - \frac{1}{2} \cdot \frac{3}{5}\right) = \frac{3}{16}. \end{aligned}$$

Finally, we complete the puzzle with $\mathcal{T}_{1,4,1}$, the last piece. We have

$$\begin{aligned} g^{\text{du}}(1/2, 1/2) &= \frac{1}{2}(\alpha_2(1/2, 1/2) + \alpha_{13}(1/2, 1) + \alpha_{31}(1/2, 1/2) + \alpha_{141}(1/2, 1)) \\ &= \frac{1}{2}\left(\frac{2}{3} + \frac{1}{3} + \frac{1}{3} + \frac{2}{3}\right) = 1, \\ g^{\text{du}}(2/7, 4/7) &= \frac{1}{2}(\alpha_{13}(2/7, 5/7) + \alpha_{123}(2/7, 1) + \alpha_{141}(2/7, 5/7) + \alpha_{1241}(2/7, 1)) \\ &= \frac{1}{2}\left(\frac{19}{30} + \frac{11}{30} + \frac{11}{30} + \frac{19}{30}\right) = 1, \end{aligned}$$

and, by symmetry, $g^{\text{du}}(4/7, 2/7) = 1$. We see that g^{du} takes at $(1/2, 1/2)$, $(2/7, 4/7)$ and $(4/7, 2/7)$ the value 1, equal to the kernel, as needed, since these are interior points. This completes the proof of the proposition. \square

6.7. Completion of the proof of Theorem 1. We first notice that \mathfrak{G} can be obtained as a particular case of the formula for \mathfrak{G}_i , with $i = 3$. Moreover, the right-hand side of (32) can also be obtained if we put $i = 3$ into the generic terms of the sums on

the right-hand side of (31). Thus, we employ Propositions 1 and 2 to obtain the sum $g(x_0, y_0) = g^{\text{du}}(x_0, y_0) + h^{\text{u}}(x_0, y_0)$, which completes the proof of Theorem 1.

7. PROOF OF THEOREM 2

Let $\mathcal{I} \subseteq [0, 1]$ be fixed and let $a'/q', a''/q''$ be consecutive fractions in \mathfrak{F}_Q . Let us first see how one translates the condition $a'/q' \in \mathcal{I}$ in terms of the two variables q', q'' . Since $a''q' - a'q'' = 1$, it follows that $a' \equiv -\overline{q''} \pmod{q'}$. Here $\overline{q''}$ is the representative from $[0, q' - 1]$ of the inverse of q'' modulo q' . Then, one immediately derives that $a'/q' \in \mathcal{I}$ if and only if $q' - \overline{q''} \in q'\mathcal{I}$. Similarly, we get that $a''/q'' \in \mathcal{I}$ if and only if $\overline{q'} \in q''\mathcal{I}$, but here the inverse is taken modulo q'' . We remark that one of these conditions is almost redundant, since $a'/q' \in \mathcal{I}$ ensures that $a''/q'' \in \mathcal{I}$, also, except for at most one pair q', q'' , and conversely. Then, imposing only one of these two conditions, one may neglect this at most one term in the corresponding computations below, and absorb it in the error term.

The proof of Theorem 2 follows the same steps from the beginning of the proof of Theorem 1. We have to find the ratio of the number of elements in the set

$$\mathcal{B}_Q^{\mathcal{I}}(r) = \left\{ (q', q'') \in \mathbb{N}^2 : \begin{array}{l} 1 \leq q', q'' \leq Q, \gcd(q', q'') = 1, q' + q'' > Q, \overline{q'} \bmod q'' \in q''\mathcal{I} \\ q' \text{ even, } q'' \text{ odd; } \mathbf{k}(q', q'') \in \mathcal{A}(r), (q', q^{\mathcal{L}}(r)) \in Q \cdot \square \end{array} \right\},$$

and the cardinality of $\mathfrak{F}_{Q, \text{even}}^{\mathcal{I}}$. Then the turning point is the analogue of relation (25), which becomes

$$\iint_{\square_{\eta}(x_0, y_0)} g_r^{\mathcal{I}}(x, y) dx dy = \lim_{Q \rightarrow \infty} \frac{\#\mathcal{B}_Q^{\mathcal{I}}(r)}{\#\mathfrak{F}_{Q, \text{even}}^{\mathcal{I}} - 1}. \quad (33)$$

This will complete the proof, provided we show that $\#\mathcal{B}_Q^{\mathcal{I}}(r) \sim |\mathcal{I}| \cdot \#\mathcal{B}_Q(r)$ and $\mathfrak{F}_{Q, \text{even}}^{\mathcal{I}} \sim |\mathcal{I}| \cdot \#\mathfrak{F}_{Q, \text{even}}$, as $Q \rightarrow \infty$. One should observe that, with the notations from Section 5, we have $\mathcal{B}_Q(r) = \mathcal{B}_Q^{[0, 1]}(r)$.

To proceed, we estimate $\mathfrak{F}_{Q, \text{even}}^{\mathcal{I}}$.

Lemma 4. *For any subinterval $\mathcal{I} \subseteq [0, 1]$, we have*

$$\mathfrak{F}_{Q, \text{even}}^{\mathcal{I}} = \frac{|\mathcal{I}|Q^2}{\pi^2} + O(Q^{3/2+\varepsilon}), \quad (34)$$

Proof. The cardinality of $\mathfrak{F}_{Q,\text{even}}^{\mathcal{I}}$ can be written as

$$\begin{aligned} \#\mathfrak{F}_{Q,\text{even}}^{\mathcal{I}} &= \#\left\{(q', q'') \in \mathbb{N}^2: \begin{array}{l} 1 \leq q', q'' \leq Q, \gcd(q', q'') = 1, q' + q'' > Q, \\ q' \text{ even}, q'' \text{ odd}, \overline{q'} \bmod q'' \in q''\mathcal{I} \end{array}\right\} \\ &= \sum_{\substack{1 \leq q \leq Q \\ q \text{ odd}}} \#\{x \in (Q - q, Q]: \gcd(x, q) = 1, x \text{ even}, \overline{x} \bmod q \in q\mathcal{I}\}. \end{aligned} \quad (35)$$

Notice that the condition $\overline{q'} \in q''\mathcal{I}$ introduces randomness in the positioning of points from the right hand side of (35). In order to estimate the terms added in the sum, we write them using exponential sums, separate the main term, and employ classical bounds for Kloosterman sums (see [13], [22]). Thus, they can be written as

$$\begin{aligned} \sum_{\substack{Q-q < x \leq Q \\ \gcd(x, q) = 1 \\ x \text{ even}}} \sum_{y \in q\mathcal{I}} \frac{1}{q} \sum_{k=1}^q e\left(k \frac{y - \overline{x}}{q}\right) &= \\ &= \frac{\phi(q)}{q} \cdot \frac{q}{2} \cdot q|\mathcal{I}| + \frac{1}{q} \sum_{k=1}^{q-1} \sum_{y \in q\mathcal{I}} e\left(k \frac{y}{q}\right) \sum_{\substack{Q-q < x \leq Q \\ \gcd(x, q) = 1 \\ x \text{ even}}} e\left(\frac{-k\overline{x}}{q}\right) \\ &= \frac{\phi(q)}{q} \cdot \frac{q}{2} \cdot q|\mathcal{I}| + O\left(q^{1/2+\varepsilon}\right), \end{aligned}$$

in which $\phi(q)$ is Euler's totient function. Then, substituting in (35), we obtain

$$\mathfrak{F}_{Q,\text{even}}^{\mathcal{I}} = \frac{|\mathcal{I}|}{2} \sum_{\substack{1 \leq q \leq Q \\ q \text{ odd}}} \frac{\phi(q)}{q} \cdot q + O\left(Q^{3/2+\varepsilon}\right), \quad (36)$$

It remains to estimate the sum from (36). This is

$$\begin{aligned} \sum_{\substack{1 \leq q \leq Q \\ q \text{ odd}}} \frac{\phi(q)}{q} \cdot q &= \sum_{\substack{1 \leq q \leq Q \\ q \text{ odd}}} \sum_{d|q} \frac{\mu(d)}{d} \cdot q = \sum_{\substack{d=1 \\ d \text{ odd}}}^Q \frac{\mu(d)}{d} \sum_{\substack{q_1=1 \\ q_1 \text{ odd}}}^{Q/d} d \cdot q_1 \\ &= \sum_{\substack{d=1 \\ d \text{ odd}}}^Q \frac{\mu(d)}{d} \cdot \frac{1}{2d} \int_1^Q t \, dt + O(Q \log Q) \\ &= \frac{2Q^2}{\pi^2} + O(Q \log Q), \end{aligned} \quad (37)$$

since, via the Euler product, we find that

$$\begin{aligned} \sum_{\substack{d=1 \\ d \text{ odd}}}^Q \frac{\mu(d)}{d^2} &= \prod_{p \geq 3} \left(1 - \frac{1}{p^2}\right) + O(Q) \\ &= \frac{6}{\pi^2} \left(1 - \frac{1}{2^2}\right)^{-1} + O(Q). \end{aligned}$$

Now, the lemma follows by inserting the estimation (37) on the right-hand side of (36). \square

We remark that the size of the error term in (34) may be slightly lowered, but this is not essential for our needs.

Next, for any $\Omega \subset \mathbb{R}^2$ and $\mathcal{I} \subseteq [0, 1]$, we denote

$$N_{\text{even,odd}}^{\mathcal{I}}(\Omega) := \# \left\{ (x, y) \in \Omega \cap \mathbb{Z}^2 : x \text{ even, } y \text{ odd, } \gcd(x, y) = 1, \bar{x} \bmod y \in y\mathcal{I} \right\}.$$

Then, in the spirit of Lemma 4, we get, more generally, the following result.

Lemma 5. *Let $R > 0$ and $\Omega \subseteq [0, R] \times [0, R]$ be a convex domain. Then, we have*

$$N_{\text{even,odd}}^{\mathcal{I}}(\Omega) = |\mathcal{I}| \cdot N_{\text{even,odd}}(\Omega) + O(R^{3/2+\varepsilon}).$$

Proof. The proof follows the first part of the proof of Lemma 4. We have:

$$\begin{aligned} N_{\text{even,odd}}^{\mathcal{I}}(\Omega) &= \# \left\{ (q_1, q_2) \in \Omega \cap \mathbb{N}^2 : \begin{array}{l} \gcd(q_1, q_2) = 1, \ q_1 \text{ even, } q_2 \text{ odd,} \\ \overline{q_1} \bmod q_2 \in q_2 \mathcal{I} \end{array} \right\} \\ &= \sum_{\substack{1 \leq q \leq R \\ q \text{ odd}}} \# \left\{ x \in \mathfrak{I}(q) : \gcd(x, q) = 1, \ x \text{ even, } \bar{x} \bmod q \in q\mathcal{I} \right\}, \end{aligned} \tag{38}$$

in which $\mathfrak{I}(q) := \Omega \cap \{y = q\}$. Evaluating the size of the terms here, using again the same estimates for Kloosterman sums, we find that they are equal to

$$\sum_{\substack{x \in \mathfrak{I}(\Omega) \\ \gcd(x, q)=1 \\ x \text{ even}}} \sum_{y \in q\mathcal{I}} \frac{1}{q} \sum_{k=1}^q e\left(k \frac{y - \bar{x}}{q}\right) = \frac{\phi(q)}{q} \cdot \frac{|\mathfrak{I}(\Omega)|}{2} \cdot |\mathcal{I}| + O(q^{1/2+\varepsilon}). \tag{39}$$

The required result follows by (38) and (39), and the fact that

$$N_{\text{even,odd}}(\Omega) = \sum_{\substack{1 \leq q \leq R \\ q \text{ odd}}} \frac{\phi(q)}{q} \cdot \frac{|\mathfrak{I}(\Omega)|}{2} + O(R).$$

\square

Now we have all the tools needed to complete the proof of Theorem 2. Since $\mathfrak{F}_{Q,\text{even}} = Q^2/\pi^2 + O(Q \log Q)$, by Lemma 4 we find that $\#\mathfrak{F}_{Q,\text{even}}^{\mathcal{I}} = |\mathcal{I}| \cdot \#\mathfrak{F}_{Q,\text{even}} + O(Q^{3/2+\varepsilon})$. On the other hand, using notations from Section 5, we find that $\mathcal{B}_Q^{\mathcal{I}}(r) = N_{\text{even,odd}}^{\mathcal{I}}(\Omega_Q(r))$, where $\Omega_Q(r) = \Omega_Q(x_0, y_0, \eta)(r)$ is given by

$$\Omega_Q(r) = \left\{ (x, y) \in \mathbb{R}^2 : \begin{array}{l} 1 \leq x, y \leq Q, \ x + y > Q, \\ \mathbf{k}(x, y) \in \mathcal{A}(r), \ (x, x_r^{\mathcal{L}}(x, y)) \in Q \cdot \square \end{array} \right\}.$$

The set $\Omega_Q(r)$ is in general not convex, but it is a finite union of boundedly many convex sets, as $Q \rightarrow \infty$, the number of these convex sets depending on the given point $(x_0, y_0) \in [0, 1] \times [0, 1]$. Then, by Lemma 5, it follows that

$$\#\mathcal{B}_Q^{\mathcal{I}}(r) = |\mathcal{I}| \cdot \#\mathcal{B}_Q(r) + O_{(x_0, y_0)}(Q^{3/2+\varepsilon}),$$

concluding the proof of Theorem 2.

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CC: INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, P.O. BOX 1-764, BUCHAREST 70700, ROMANIA.

E-mail address: cristian.cobeli@imar.ro

AZ: INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, P.O. BOX 1-764, BUCHAREST 70700, ROMANIA.

Current address: AZ: Department of Mathematics, University of Illinois at Urbana-Champaign, Altgeld Hall, 1409 W. Green Street, Urbana, IL, 61801, USA.

E-mail address: zaharesc@math.uiuc.edu